Day-5 20/12/24

Last time

1(a) • $\mathrm{char}(M)$, $M \in \mathcal{O}$

$\mathrm{char}(M(\lambda))$ are $\mathbb{C}$-linear independent

• $K(\mathcal{O})$ is $\mathbb{Z}$-module

• $K(\mathcal{O}) = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[L(\lambda)] = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)]$

For $\theta = \theta_\lambda = W \cdot \lambda \in \mathfrak{h}^*/(W, \cdot)$

• $K(\mathcal{O})_\theta = \bigoplus_{w \in W} \mathbb{Z}[L(w \cdot \lambda)]$

$\Rightarrow K(\mathcal{O}) = \bigoplus_{\theta \in \mathfrak{h}^*/(W, \cdot)} K(\mathcal{O})_\theta$ — $(*)$

• showed: $K(\mathcal{O})_\theta = \bigoplus_{w \in W} \mathbb{Z}[M(w \cdot \lambda)]$

using the decomposition matrix

$D = \left([M(\lambda_i) : L(\lambda_j)]\right)_{i,j=1}^{s}$, by

proving $D = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_s(\mathbb{Z})$ up to reindexing.

1(b) $\boxed{\text{Cor}}$ char $L(\lambda)$, $\lambda \in \mathfrak{h}^*$ are (also) $\mathbb{C}$-linear independent in $V_{\mathfrak{g}}$

Today:- show that $\circledast$ holds on the level of categories/$\mathfrak{g}$-representations

Recall: If $M(\lambda) \twoheadrightarrow V \neq 0$ and $[V : L(\mu)] > 0$

then $\chi_\lambda = \chi_\mu \iff \mu \in W \cdot \lambda$

(2) Goal:- $\mathcal{O} = \bigoplus_{\theta \in \mathfrak{h}^*/(W, \cdot)} \mathcal{O}_\theta$; where

2(a) Def^n : Given $\theta = \theta_\lambda \in \mathfrak{h}^*/(W, \cdot)$

define the subcategory/block $\mathcal{O}_\theta$ to be the full subcategory of $\mathcal{O}$ of those objects s.t. $\underbrace{[M : L(\mu)]}_{\text{Jordan-Holder factors}} > 0 \Rightarrow \mu \in \theta_\lambda = W \cdot \lambda$

and

2(b) $\bigoplus \mathcal{O}_\theta$ means: if $\theta_1 \neq \theta_2$

and $M_j \in \mathcal{O}_{\theta_j}$ for $j = 1, 2$.

then

- $\mathrm{Hom}_{\mathcal{O}}(M_1, M_2) = 0$

- $\mathrm{Ext}^1_{\mathcal{O}}(M_1, M_2) = 0 \hookrightarrow$ any short exact sequence in $\mathcal{O}$

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

splits : $M \cong M_2 \oplus M_1$

③ "Baby case 1"

Proposition: Say $\lambda, \mu \in \mathfrak{h}^*$ with $\mu \notin W \bullet \lambda$

Also fix highest weight modules (non-zero):

$(V, V^\vee)$

$$M(\lambda) \twoheadrightarrow V \quad , \quad M(\mu) \twoheadrightarrow U$$

then

$$\mathrm{Hom}_{\mathcal{O}}(V, U) = \mathrm{Ext}^1_{\mathcal{O}}(V, U) = 0$$

proof:

3(a) Fix $0 \neq v_\lambda \in V_\lambda$,

say $0 \neq \varphi \in \mathrm{Hom}_{\mathcal{O}}(V, U)$

$\Rightarrow \varphi(v_\lambda)$ is maximal vector in U_λ

$\Rightarrow$ on $\varphi(v_\lambda)$, $Z(\mathcal{U}\mathfrak{g})$ acts by χ_λ

$\because \varphi(v_\lambda) \in U$, $Z(\mathcal{U}\mathfrak{g})$ acts by χ_μ

But $\chi_\lambda \neq \chi_\mu$ by Harish chandra thm;
we get contradiction.

3(b) Next, say $0 \longrightarrow \underbrace{U}_{\substack{\text{has} \\ \text{h.w } \mu}} \longrightarrow M \xrightarrow{\pi} \underbrace{V}_{\substack{\text{has h.w.} \\ \lambda}} \longrightarrow 0$ in $\mathcal{O}$

- Fix $z_0 \in Z(\mathcal{U}\mathfrak{g})$: $\chi_\lambda(z_0) \neq \chi_\mu(z_0)$
- Note: $\dim M_\lambda = 1 + \dim U_\lambda$

Moreover, $\big(z_0 - \chi_\mu(z_0)\big)(M_\lambda) \neq 0$

$$\left(\because \pi\big(z_0 - \chi_\mu(z_0)\big)M_\lambda = \big(z_0 - \chi_\mu(z_0)\big)\underbrace{\pi(M_\lambda)}_{V_\lambda} = \big(\chi_\lambda(z_0) - \chi_\mu(z_0)\big).V_\lambda \right.$$
$$\left. = V_\lambda \neq 0 \right)$$

But $\big(z_0 - \chi_\mu(z_0)\big)(U_\lambda) = 0$

$\Rightarrow \big(\underset{\substack{\cap \\ (\mathcal{U}\mathfrak{g})}}{z_0} - \chi_\mu(z_0)\big)(M_\lambda)$ is 1-dim subspace of M_λ,

say $\big(z_0 - \chi_\mu(z_0)\big)(M_\lambda) = \mathbb{C}.m_\lambda$ $\quad (\because V_\lambda = \mathbb{C}\, m_\lambda)$

$\Rightarrow \big(z_0 - \chi_\mu(z_0)\big)m_\lambda = d' m_\lambda \Rightarrow z_0 \cdot m_\lambda = d \cdot m_\lambda$

3(c) so $m_\lambda \xmapsto{\pi} v_\lambda$

$\Rightarrow m_\lambda \notin \ker\pi = U$

$\Rightarrow M_\lambda = U_\lambda \oplus \mathbb{C}.m_\lambda$

3(d) claim :- $(z_0 - \chi_\lambda(z_0)).m_\lambda = 0$

By 3(b), $z_0 m_\lambda = d.m_\lambda$, $\exists\, d \in \mathbb{C}$

$\Rightarrow \pi(z_0.m_\lambda) = \pi(d.m_\lambda)$

$\Rightarrow z_0.v_\lambda = d.v_\lambda$

Thus $z_0.v_\lambda = \chi_\lambda(z_0).v_\lambda = d.v_\lambda$

$\Rightarrow d = \chi_\lambda(z_0)$ $\blacksquare$

3(e) let $\tilde{V} := \mathcal{U}_{\mathfrak{g}}.m_\lambda \underset{\mathfrak{g}\text{-mod}}{\subseteq} M$

say $m \in U \cap \tilde{V}$

$\Rightarrow \chi_\mu(z_0).m \overset{\in U}{=\!=} z_0.m \overset{\in \tilde{V}}{=\!=} \chi_\lambda(z_0).m$

$\Rightarrow m = 0$

i.e. $U \cap \tilde{V} = \{0\}$

3(f) So $\boxed{U \oplus \tilde{V} \subseteq M}$ — (**)

$\Rightarrow$ char U + char $\tilde{V}$ $\leq$ char M

Since $0 \to U \longrightarrow M \xrightarrow{\pi} V \longrightarrow 0$ in $\mathcal{O}$

$0 \neq m_\lambda \longmapsto v_\lambda \neq 0$

so $\tilde{V} = \mathcal{U}\mathfrak{g} . m_\lambda \overset{\pi}{\twoheadrightarrow} \mathcal{U}\mathfrak{g} . v_\lambda = V$

$\Rightarrow$ char(U) + char$(\tilde{V})$ $\geq$ char(U) + char(V)

$=$ char(M)

Hence char(U) + char$(\tilde{V})$ = char(M), which together with (**) implies

- $U \oplus \tilde{V} \subseteq M$ is an equality
- char $\tilde{V}$ = char V, and $\tilde{V} \twoheadrightarrow V$

so $\tilde{V} \cong V$.

So, $M = U \oplus \tilde{V} \cong U \oplus V$ ▨

(4) "Baby Case 2: $\mathfrak{g} = \mathfrak{sl}_2$"

$M = M_{2,\lambda} = \mathcal{U}\mathfrak{g} . 1 / \langle \mathcal{U}\mathfrak{g} \cdot e^2 + \mathcal{U}\mathfrak{g}\,(h - \lambda(h)) \rangle$

We saw

$$0 \longrightarrow \underset{\substack{\| \\ \mathcal{U}\mathfrak{g}\cdot e}}{M(\lambda+\alpha)} \longrightarrow M \longrightarrow \underset{\substack{\| \\ \mathcal{U}\mathfrak{g}\cdot \bar{1}}}{M(\lambda)} \longrightarrow 0$$

4(a): Calculation: Solve for λ: $s_\alpha\cdot\lambda = \lambda+\alpha$

(Ans: $s_\alpha(\lambda+\rho) = \lambda+\alpha+\rho$
$\iff s_\alpha(\lambda) = \lambda+2\alpha \iff \lambda = -\alpha$

4(b): So, if $\lambda = -\alpha$; $M(\lambda+\alpha), M(\lambda)$ have the same central character.

Hence $M(\lambda+\alpha), M(\lambda) \in \mathcal{O}_{\theta_0}$

and in this case, the short exact sequence for $M_{2,\lambda}$ doesn't split.

4(c) For all other λ, $\chi_\lambda \neq \chi_{\lambda+\alpha}$,

so by "Baby case 1", this short exact sequence splits:

$$M_{2,\lambda} \cong \underset{\substack{\| \\ \mathcal{U}\mathfrak{g}\cdot e}}{M(\lambda+\alpha)} \oplus \underset{\substack{\| \\ \mathcal{U}\mathfrak{g}(fe-(\lambda+\alpha)(h)\cdot 1)}}{M(\lambda)}$$

$\left[\because\ e\cdot(fe-(\lambda+\alpha)(h)\cdot 1) = \cancel{fe^2} + he - (\lambda+\alpha)(h)\cdot 1\right.$

$= 0$ in $M_{2,\lambda}$ $\boxed{\lambda(h)=-2 \text{ as } \lambda=-\alpha}$

$\left.\Rightarrow fe-(\lambda+\alpha)(h)\cdot 1 \text{ is highest weight vector}\right]$ $\boxtimes$

Remark:- If $\lambda=-\alpha$ in $M_{2,\lambda}$, then

$(z-\chi_\lambda(z))\cdot M_{2,\lambda}$ is non-zero for some $z\in Z(U(sl_2))$.

Check

$z = \underbrace{h^2+2h+4fe}_{\text{casimir element}} \in Z(U(sl_2))$ i.e; z_0 commutes with e, f, h.

$\Rightarrow\ \chi_0(z) = 0(h^2+2h) = 0$

and

$$0 \longrightarrow \underbrace{M(0)}_{\lambda+\alpha} \longrightarrow M \longrightarrow \underbrace{M(-\alpha)}_{\lambda} \longrightarrow 0$$

(χ_0 is character of $M(0)$
so $z-\chi_0(z)=0\ \forall\ z\in M(0)$)

so $z-\chi_0(z):\ M \longrightarrow M(0) \longrightarrow 0$

and $(z-\chi_0(z))\cdot\overline{1} = 4fe\cdot 1 + \underbrace{(h^2+2h)\cdot 1}_{\overset{\shortparallel}{0}}$ $\left(\because (-2)^2+2(-2)=0\right]$

$\neq 0$

$\Rightarrow (z-\chi_0(z))$ doesn't kill $M_{2,\lambda}$

but $(z - \chi_0(z))^2$ does; where z = Casimir element. $\blacksquare$

4(d). This phenomenon happens for all semi-simple $\mathfrak{g}$: $\exists$ non-split extensions in $\mathcal{O}_\theta$'s.

This motivates:

4(e). **Definition**: Given $\mathfrak{g}$ = semi-simple over $\mathbb{C}$ and $\theta \in \mathfrak{h}^*/(W,\cdot)$ define $\mathcal{O}'_\theta$ to be the full subcategory of $\mathcal{O}$ with all objects:

$$\left\{ M \in \mathcal{O} : \begin{array}{c} \forall\, z \in Z(U_\mathfrak{g}) \\ \exists\, k > 0 \text{ s.t.} \\ (z - \chi_\theta(z))^k (M) = 0 \end{array} \right\}$$

5(a) Recall

$$\mathcal{O}_\theta := \{ M \in \mathcal{O} : [M : L(\mu)] > 0 \Rightarrow \chi_\mu = \chi_\theta \}$$

Thm ① $\mathcal{O}_\theta = \mathcal{O}_\theta' \quad \forall\, \theta \in \mathfrak{h}^*/(W,\cdot)$.

This is an abelian finite length subcategory of $\mathcal{O}$.

② $$\mathcal{O} = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\theta$$

i.e., $\forall M \in \mathcal{O}$, $M = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\cdot)} M_\theta$ with all $M_\theta \in \mathcal{O}_\theta$

③ Let $\theta_1 \neq \theta_2$,

$M_j \in \mathcal{O}_{\theta_j}$ ~~for~~ $j = 1, 2$

Then $\mathrm{Hom}_{\mathcal{O}}(M_1, M_2) = \mathrm{Ext}^1_{\mathcal{O}}(M_1, M_2) = 0$

Cor.
$$\begin{array}{ccc} K(\mathcal{O}) & = & \bigoplus_\theta K(\mathcal{O}_\theta) \\ \| & & \| \\ & \bigoplus_\theta (K(\mathcal{O}))_\theta & \end{array}$$

$\longleftarrow$ From last lecture

5(b) proof of ①

Let $M \in \mathcal{O}$, and say

$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M \qquad (n = \ell(M))$

s.t. $M_{j+1}/M_j \cong L(\mu_j) \quad \forall \ 0 \leq j < n$

(I) Say $M \in \mathcal{O}_\theta \Rightarrow$ all $\mu_j \in \theta = (W \cdot \mu_j)$

then $\forall z \in Z(\mathcal{U}\mathfrak{g})$

$(z - \chi_\theta(z))(L(\mu_j)) = 0$

$$\Rightarrow \left(z - \chi_\theta(z) \right) : \quad M_{j+1} \longrightarrow M_j \quad \forall j$$

so

$$M = M_n \xrightarrow{z-\chi_\theta(z)} M_{n-1} \xrightarrow{z-\chi_\theta(z)} M_{n-2} \to \cdots \xrightarrow{z-\chi_\theta(z)} M_1 \to M_0 = 0$$

so $\left(z - \chi_\theta(z) \right)^{l(M)} : M \longrightarrow 0 \quad \forall M \in \mathcal{O}_\theta$

$\forall z \in Z(\mathcal{U}_\mathfrak{g})$

$$\Rightarrow \mathcal{O}_\theta \subseteq \mathcal{O}'_\theta$$

II conversely, say $M \in \mathcal{O}'_\theta \setminus \mathcal{O}_\theta$

Recall

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M \quad (n = l(M))$$

s.t. $M_{j+1}/M_j \cong L(\mu_j) \quad \forall \; 0 \le j < n$

so $\exists \; j \in [0, n-1]$ s.t. $\chi_{\mu_j} \neq \chi_\theta \quad (\mu_j \notin \theta)$

Take
- $z_0 \in Z(\mathcal{U}_\mathfrak{g})$ s.t. $\chi_{\mu_j}(z_0) \neq \chi_\theta(z_0)$
- and $m_{\mu_j} \in M_{j+1} \subseteq M$ whose image generates $M_{j+1}/M_j \cong L(\mu_j)$.

$v_{\mu_j} \in$

Now write the short exact sequence

$$0 \to M_j \to M_{j+1} \xrightarrow{\pi} L(\mu_j) \to 0$$

and compute $\forall\, k \geq 0$:

$$\pi\left[(z_0 - \chi_\theta(z_0))^k \cdot m_{\mu_j}\right] = (z_0 - \chi_\theta(z_0))^k \pi(m_{\mu_j})$$

$$= (\underbrace{\chi_{\mu_j}(z_0) - \chi_\theta(z_0)}_{\in \mathbb{C}^* \text{ so } \neq 0})^k \cdot v_{\mu_j} \neq 0$$

$\Rightarrow (z_0 - \chi_\theta(z_0))^k$ doesn't kill M_{j+1}, hence not M, which is contradiction.

$$\Rightarrow \mathcal{O}'_\theta \subseteq \mathcal{O}_\theta .$$

■

5(c). $K(\mathcal{O}_\theta) = K(\mathcal{O})_\theta = \bigoplus_{\lambda \in \theta} \mathbb{Z}[L(\lambda)]$

"So" $\mathcal{O}_\theta$ is a finite length abelian category.

⑥ proof of part (3) of thm 5(a):

6(a) • Key tools to show Hom's and Ext's $= 0$

are the
- long exact seq. of Ext's
- weaving lemma
- Zig-Zag lemma
- Snake lemma

6(b) <u>Recall</u>: A sequence

$$\cdots \longrightarrow A \xrightarrow{d} B \xrightarrow{d'} C \longrightarrow \cdots$$

in an abelian category, is
- exact at B if $\ker(d') = \operatorname{im}(d)$
- exact if it is exact at all objects in it.

6(c) **Snake lemma**: say

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] \arrow[dll, bend left, "\text{snake}"] & D \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

has exact rows.

Then $\exists$ exact sequence

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \longrightarrow \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

6(d) **Long exact sequences - 1**

In an abelian category $\mathcal{O}$, applying $\operatorname{Hom}_{\mathcal{O}}(N, -)$

to a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

yields a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}}(N, M') \to \mathrm{Hom}_{\mathcal{O}}(N,M) \to \mathrm{Hom}_{\mathcal{O}}(N,M'')$$

$$\to \mathrm{Ext}^1_{\mathcal{O}}(N,M') \to \mathrm{Ext}^1_{\mathcal{O}}(N,M) \to \mathrm{Ext}^1_{\mathcal{O}}(N,M'')$$

$$\to \mathrm{Ext}^2_{\mathcal{O}}(N,M') \to \mathrm{Ext}^2_{\mathcal{O}}(N,M) \to \mathrm{Ext}^2_{\mathcal{O}}(N,M'')$$

$$\to \cdots \qquad \cdots \qquad \to \cdots$$

6(e) Long exact sequences - 2

In $\mathcal{O}$, applying $\mathrm{Hom}_{\mathcal{O}}(-,N)$ instead yields the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}}(M'',N) \to \mathrm{Hom}_{\mathcal{O}}(M,N) \to \mathrm{Hom}_{\mathcal{O}}(M',N)$$

$$\to \mathrm{Ext}^1_{\mathcal{O}}(M'',N) \to \mathrm{Ext}^1_{\mathcal{O}}(M,N) \to \mathrm{Ext}^1_{\mathcal{O}}(M',N)$$

$$\to \mathrm{Ext}^2_{\mathcal{O}}(M'',N) \to \cdots$$

6(f) Application 1

$\mathrm{Hom}_{\mathcal{O}}(M_{\theta_1}, M_{\theta_2}) = 0$ when $\theta_1 \neq \theta_2 \in \mathfrak{h}^*/(W,\cdot)$, $M_{\theta_j} \in \mathcal{O}_{\theta_j}$.

proof :-

First let M_{θ_1} = simple, say $L(\lambda)$, $\lambda \in \theta_1$

we claim : $\mathrm{Hom}(L(\lambda), M) = 0 \ \forall M \in \mathcal{O}_{\theta_2}$ by induction on $\ell(M) \geq 0$

- $\ell = 0$, holds trivially.
- $\ell = 1$, $\mathrm{Hom}_{\mathcal{O}}(L(\lambda), L(\mu)) = 0 \ \forall \mu \in \theta_2 \neq \theta_1 \ni \lambda$ by "Baby case 1".
- Induction step: let $\ell(M) = \ell$

say
$$0 \to M_{\ell-1} \to M \to L(\mu) \to 0 \quad \text{in } \mathcal{O}_{\theta_2}$$
and we know the claim for $M_{\ell-1}$, $L(\mu)$.

Applying $\mathrm{Hom}_{\mathcal{O}}(L(\lambda), -)$ to this short exact seq, we get

$$0 \to \underbrace{\mathrm{Hom}(L(\lambda), M_{\ell-1})}_{\substack{\| \\ 0 \text{ (by induction step)}}} \to \mathrm{Hom}(L(\lambda), M) \to \underbrace{\mathrm{Hom}(L(\lambda), L(\mu))}_{\substack{\| \\ 0}} \to \cdots$$

$\Rightarrow \operatorname{Hom}(L(\lambda), M) = 0 \quad \forall \ \lambda \in \Theta_1, M \in \mathcal{O}_{\Theta_2}$

Next, we show the claim in general, i.e.,

claim 2: Fix $M_{\Theta_2} \in \mathcal{O}_{\Theta_2}$. Then

$$\operatorname{Hom}_{\mathcal{O}}(M, M_{\Theta_2}) = 0 \quad \forall \ M \in \mathcal{O}_{\Theta_1}$$

by induction on $\ell(M) \geq 0$

- $\ell = 0$, holds
- $\ell = 1 \rightsquigarrow$ as shown above
- Induction step:

say $0 \to M_{\ell-1} \to M \to L(\lambda) \to 0$ in $\mathcal{O}_{\Theta_1}$

applying $\operatorname{Hom}(-, M_{\Theta_2})$; we get

$$0 \to \underbrace{\operatorname{Hom}(L(\lambda), M_{\Theta_2})}_{\substack{\| \\ 0 \ (\text{by } \ell=1)}} \to \operatorname{Hom}(M, M_{\Theta_2}) \to \underbrace{\operatorname{Hom}(M, M_{\ell-1})}_{\substack{\| \\ 0 \ (\text{by induction hypothesis})}} \to \cdots$$

so, $\operatorname{Hom}(M, M_{\Theta_2}) = 0 \quad \forall \ M \in \mathcal{O}_{\Theta_1}, \ M_{\Theta_2} \in \mathcal{O}_{\Theta_2}$ $\boxtimes$

6(g) Application 2

claim 1: $\operatorname{Ext}^1(L(\lambda), M) = 0 \quad \forall \ \lambda \in \Theta_1, \ M \in \mathcal{O}_{\Theta_2}$

by induction on $\ell(M) \geq 0$

- $l=0$, holds trivially
- $l=1$, $\mathrm{Ext}^1(L(\lambda), L(\mu))=0$ by "Baby Case 1"
- <u>Induction step</u>:

say $0 \to M_{l-1} \to M \to L(\mu) \to 0$ in $\mathcal{O}_{\theta_2}$

& we know the claim for $M_{l-1}, L(\mu)$.

Applying $\mathrm{Hom}_{\mathcal{O}}(L(\lambda), -)$ to this short exact seq.

we get,

$$0 \to \underset{0}{\mathrm{Hom}(L(\lambda), M_{l-1})} \to \underset{0}{\mathrm{Hom}(L(\lambda), M)} \to \underset{0}{\mathrm{Hom}(L(\lambda), L(\mu))}$$

$$\to \underbrace{\mathrm{Ext}^1(L(\lambda), M_{l-1})}_{\overset{\text{induction}}{=}\,0} \to \mathrm{Ext}^1(L(\lambda), M) \to \underbrace{\mathrm{Ext}^1(L(\lambda), L(\mu))}_{=0} \to \cdots$$

$$\Rightarrow \mathrm{Ext}^1(L(\lambda), M)=0 \quad \forall \lambda \in \theta_1, \ M \in \mathcal{O}_{\theta_2}$$

Next, we show the claim in general, i.e.

Claim 2: Fix $M_{\theta_2} \in \mathcal{O}_{\theta_2}$. Then $\mathrm{Ext}^1_{\mathcal{O}}(M, M_{\theta_2})=0 \ \forall M \in \mathcal{O}_{\theta_1}$, by induction on $l(M) \geq 0$

- $l=0$ holds

- $l=1$, shown above
- <u>Induction step</u>:

say $0 \to M_{l-1} \to M \to L(\lambda) \to 0$ in $\mathcal{O}_\theta$,

Applying $\mathrm{Hom}(-, M_{\theta_2})$, we get

$$0 \to \overset{0}{\overset{\shortparallel}{\mathrm{Hom}(L(\lambda), M_{\theta_2})}} \to \overset{0}{\overset{\shortparallel}{\mathrm{Hom}(M, M_{\theta_2})}} \to \overset{0}{\overset{\shortparallel}{\mathrm{Hom}(M_{l-1}, M_{\theta_2})}}$$

$$\to \underset{0}{\underset{\shortparallel}{\mathrm{Ext}^1(L(\lambda), M_{\theta_2})}} \to \mathrm{Ext}^1(M, M_{\theta_2}) \to \underset{0}{\underset{\shortparallel}{\mathrm{Ext}^1(M_{l-1}, M_{\theta_2})}} \text{ by induction}$$

$$\to \cdots$$

$$\Rightarrow \quad \mathrm{Ext}^1(M, M_{\theta_2}) = 0 \quad \forall\, M \in \mathcal{O}_\theta,\ M_{\theta_2} \in \mathcal{O}_{\theta_2}. \quad \blacksquare$$

⑦ Remains to show the part ②:

$$\mathcal{O} = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\theta$$

<u>7(a)</u> claim:- All Hom spaces in $\mathcal{O}$ are f.d.

i.e., $\dim \mathrm{Hom}_{\mathcal{O}}(M, M') < \infty \quad \forall\, M, M' \in \mathcal{O}$.

<u>Proof</u>: Say M = generated by $m_1, \ldots, m_r$,

where $m_j \in M_{\mu_j}$ $\forall j$

Any $\mathfrak{g}$-module map $\varphi : M \longrightarrow M'$

- is uniquely determined by $\{\varphi(m_j)\}_{1 \le j \le k}$

and • sends $\varphi(m_j)$ to $\underbrace{M'_{\mu_j}}_{\text{f.d.}}$ $\forall j$.

so $$\underbrace{\mathrm{Hom}_{\mathcal{O}}(M, M')}_{\mathfrak{g}\text{-mod}} \subseteq \underbrace{\bigtimes_{j=1}^{k}}_{\substack{\text{Cartesian}\\ \text{product}}} \mathrm{Hom}_{\mathfrak{h}\text{-mod}}(\mathbb{C} m_j, M')$$

$$= \bigtimes_{j=1}^{k} M'_{\mu_j}$$

$$= \text{f.d.} \quad (\because M'_{\mu_j} = \text{f.d.}) \quad \boxtimes$$

7(b) Say $0 \to V_1 \to V_2 \to \cdots \longrightarrow V_K \to 0$ is an exact sequence of vector spaces, all of which are finite-dimensional.

Then $$\underbrace{\dim V_1 - \dim V_2 + \cdots \pm \dim V_K}_{=: \text{ Euler characteristic } \chi(V_\bullet)} = 0$$

7(c) This (i.e. 7(b)) holds even if exactly one V_j is not known to be finite dim a priori.

7(d) proof of thm 5(a)'s part (2)

Claim: Every $M \in \mathcal{O} = \bigoplus M_\theta$, $\theta \in \mathfrak{h}^*/(W,\cdot)$ by induction on $l(M)$

- $l=0$ holds trivially
- $l=1$ $L(\lambda) \in \mathcal{O}_{\theta_\lambda = W\cdot\lambda}$
- Induction step: say $0 \to M_{l-1} \to M \twoheadrightarrow L(\lambda) \to 0$ in $\mathcal{O}$

Why does M split?

7(e) We use:

Prop: say $0 \to A \oplus B' \to C \to B'' \to 0$ is a short exact seq. in (abelian category) $\mathcal{C}$ of $\mathbb{C}$-vector spaces.

If $\mathrm{Ext}^1_{\mathcal{O}}(B'', A) = 0$ then $C = A \oplus B$
where $0 \to B' \to B \to B'' \to 0$ and $B \in \mathcal{O}$.

proof:- Apply $\mathrm{Hom}_{\mathcal{O}}(B'', -)$ to the seq.

- $$0 \to B' \to B' \oplus A \longrightarrow A \to 0$$, to get:

$$0 \to \mathrm{Hom}_{\mathcal{O}}(B'', B') \longrightarrow \mathrm{Hom}_{\mathcal{O}}(B'', B' \oplus A) \to \mathrm{Hom}_{\mathcal{O}}(B'', A)$$
$$\to \mathrm{Ext}^1_{\mathcal{O}}(B'', B') \xrightarrow{\varphi} \mathrm{Ext}^1_{\mathcal{O}}(B'', B' \oplus A) \to \underset{\underset{0}{\parallel}}{\mathrm{Ext}^1_{\mathcal{O}}(B'', A)} \to \cdots$$

First row is exact (check!)

$$\Rightarrow \mathrm{Ext}^1_{\mathcal{O}}(B'', B') \xrightarrow[\sim]{\varphi} \mathrm{Ext}^1_{\mathcal{O}}(B'', B' \oplus A)$$

But φ takes an extension/sequence
$$0 \to B' \to M \longrightarrow B'' \to 0$$ to the sequence
$$0 \longrightarrow B' \oplus A \longrightarrow M \oplus A \to B'' \longrightarrow 0.$$

As φ is an isomorphism, the extension C in the proposition must be of this form:
$$C = B \oplus A,$$
where $0 \to B' \to B \to B'' \to 0$ ∎

7(f) (Continuation of 7(d))

Recall $0 \to M_{\ell-1} \to M \to L(\lambda) \to 0$

$$M_{\ell-1} = M_{\theta_\lambda} \oplus \underbrace{\bigoplus_{\theta \neq \theta_\lambda} M_\theta}_{=N} \quad \text{(induction)}$$

Apply Prop 7(e) with $B'' = L(\lambda)$; $C = M$

$B' = M_{\theta_\lambda}$; $A = N$

In this case, $\mathrm{Hom}(B'', A) = 0$ by part(3) of thm

so $(\cancel{\ast\ast\ast})$ is immediate. Now by prop 7(e),

$$M = N \oplus B \quad \text{where}$$

$$0 \to M_{\theta_\lambda} \to \underset{\in \mathcal{O}_{\theta_\lambda}}{B} \to L(\lambda) \to 0$$

$\blacksquare$