# Day-6

21/12/24

①(a) In lecture-4,

$$\text{char}: K(\mathcal{O}) = \bigoplus K(\mathcal{O}_\theta) \longrightarrow \mathcal{V}_g \text{ is } 1\text{-}1$$

Q. (Tutorial): If $M \in \mathcal{O}$, $\text{char}(M) = \text{char}\, L(\lambda)$ then $M \cong L(\lambda)$, $[M] = [L(\lambda)]$

1(b) In lecture-5

$$\mathcal{O} = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\theta$$

→ Long exact sequences of Ext's

→ $\text{Hom}_{\mathcal{O}}(M, M')$ is f.d. $\forall M, M' \in \mathcal{O}$

→ What about $\text{Ext}^1(M, M')$, $\forall M, M' \in \mathcal{O}$ ?

② Thm All $\text{Ext}^1_{\mathcal{O}}(M, M')$ are f.d.

Goal

2(a) Duality in $\mathcal{O}$: We want functor $F: \mathcal{O} \to \mathcal{O}$ that "reverses morphisms".

i.e., $\varphi \in \text{Hom}_{\mathcal{O}}(M, N) \rightsquigarrow F(\varphi) \in \text{Hom}_{\mathcal{O}}(\underset{\overset{\shortparallel}{N^*}}{F(N)}, \underset{\overset{\shortparallel}{M^*}}{F(M)})$

is a contravariant functor.

Ideally, we also want

- $F$ is exact
- $F$ preserves $\mathrm{char}(-)$ [i.e. we want dimension of weight spaces to be preserved by $F$]

2(b):

Recall: Say $G$ = group, $V$ = $G$-mod

How to define $G$-rep on $V^*$?

we set $(g\cdot\varphi)(v) = \varphi(g^{-1}\cdot v)$ $\forall g\in G$, $v\in V$.

2(c) How to replicate this for an associative algebra $A$?

We want a map $i: A\to A$ s.t. the following for groups:

$$g\cdot(h\cdot\varphi)\cdot v = (h\cdot\varphi)(g^{-1}v) = \varphi(h^{-1}g^{-1}v)$$

$$((gh)\cdot\varphi)v = \varphi((gh)^{-1}\cdot v)$$

now holds for $A$-modules. That is, we want

$$(a\cdot(a'\cdot\varphi))(v) = (a'\cdot\varphi)(i(a)\cdot v) = \varphi(i(a')\, i(a)v)$$

and $((aa')\cdot\varphi)(v) = \varphi(i(aa')\cdot v)$

to be equal $\forall\ v\in V,\ \varphi\in V^*,\ a, a'\in A,$

$\therefore$ $\forall\ V = A$-modules.

This happens if $\forall\ V = A$-modules,

$$\boxed{i(aa') = i(a')\cdot i(a) \quad \forall\ a, a'\in A}.$$

2(d) This brings us to

Definition:- Given an associative $\mathbb{F}$-algebra $A$ (with unit) an anti-involution of $A$ is an $\mathbb{F}$-linear map $i : A \longrightarrow A$ s.t.

$$i(1) = 1, \quad i(aa') = i(a')\cdot i(a) \ \forall\ a, a'\in A$$

$$\text{and} \quad i^2 \equiv id_A$$

2(e) Prop$^n$: Let $\mathfrak{g}$ = semi-simple$|_{\mathbb{C}}$. Then $\exists\ i : \mathfrak{g}\to\mathfrak{g}$ s.t. (I) $i[x,y] = [i(y), i(x)]$ $\forall\ x, y\in\mathfrak{g}$

(II) $i^2 \equiv id_{\mathfrak{g}}$

(III) $i|_{\mathfrak{h}} \equiv id_{\mathfrak{h}}$

and this extends by multiplicativity and linearity to an anti-involution of $\mathcal{U}_{\mathfrak{g}}$ that fixes $\mathcal{U}(\mathfrak{h})$ pointwise.

**proof** Define $i(e_{\alpha_i}) = -f_{\alpha_i} \quad \forall\, i \in I$

$$i(f_{\alpha_i}) = -e_{\alpha_i}$$

$$i(h) = h \quad \forall\, h \in \mathfrak{h}$$

$$\Rightarrow i\,[e_{\alpha_i}, e_{\alpha_j}] \underset{\text{in } \mathcal{U}(\mathfrak{g})}{=} i(e_{\alpha_i} e_{\alpha_j} - e_{\alpha_j} e_{\alpha_i})$$

$$= i(e_{\alpha_j}) \cdot i(e_{\alpha_i}) - i(e_{\alpha_i}) \cdot i(e_{\alpha_j})$$

$$= f_{\alpha_j} f_{\alpha_i} - f_{\alpha_i} \cdot f_{\alpha_j}$$

$$= [f_{\alpha_j}, f_{\alpha_i}] = [i(e_{\alpha_j}), i(e_{\alpha_i})]$$

Note: If $i$ extends to anti-involution on $\mathcal{U}_{\mathfrak{g}}$ then $i[x,y] = i[xy - yx) = i(y) i(x) - i(x) i(y)$

$$= [i(y), i(x)]$$

Check this is compatible with relations. E.g.,

$$i\,[h, e_{\alpha_i}] = [-f_{\alpha_i}, h] = [h, f_{\alpha_i}] = -\alpha_i(h) f_{\alpha_i}$$

$$\parallel$$

$$i(\alpha_i(h) e_{\alpha_i}) = -\alpha_i(h) f_{\alpha_i}$$

& $i\,[e_{\alpha_i}, f_{\alpha_j}] = [e_{\alpha_j}, f_{\alpha_i}] = \delta_{ij} \cdot h_i$. ∎

## (3) Duality functor on $\mathcal{O}$

3(a) **Definition** :- The Harish Chandra-category $\mathcal{H}$ over semi-simple $\mathfrak{g}$ is the full subcategory of $\mathfrak{g}$-modules which are $\mathfrak{h}$-semi-simple with f.d. $\mathfrak{h}$-weight spaces.

Note $\mathcal{H} \supseteq \mathcal{O}$

3(b) Given $M = \mathfrak{g}\text{-mod} \overset{i}{\rightsquigarrow} M^* = \mathcal{U}_{\mathfrak{g}}\text{-mod}$

**Definition** : The restricted dual $F(M)$ of $M \in \mathcal{H}$ is the subspace of $M^*$ spanned by the $\mathfrak{h}$-weight vectors.

3(c) Th$^m$ Let $M \in \mathcal{H}$, and let $\{m_{\mu i}\}_{i=1}^{n_\mu}$ be a $\mathbb{C}$-basis of $M_\mu$, $\forall\, \mu \in \mathfrak{h}^*$.

$\forall\, \mu \in \mathfrak{h}^*$ basis of $M$

Define $m^*_{\lambda j}$ $(\lambda \in \mathfrak{h}^*,\ 1 \leq j \leq n_\lambda)$ "dual basis" of $M$

via $m^*_{\lambda j}(m_{\mu i}) = \delta_{\lambda\mu}\, \delta_{ji}$

(I) The $m^*_{\lambda j}$ constitute a basis of $F(M)_\lambda$

(II) so $\text{char}\, F(M) = \text{char}(M)$, so $F(M) \in \mathcal{H}$

(III) $F(F(M)) \cong M$ as $\mathfrak{g}$-module.

(IV) For all $M, M' \in \mathcal{H}$, $\text{Hom}_{\mathfrak{g}}(M, M') \cong \text{Hom}_{\mathfrak{g}}(F(M'), F(M))$

V For all $M, M' \in \mathcal{O}$

$$\text{Ext}^1_{\mathcal{H}/\mathcal{O}}(M, M') \cong \text{Ext}^1_{\mathcal{O}}(F(M'), F(M))$$

## 3(d) Partial proofs of th^m

(I) Claim 1: $m^*_{\lambda j}$ $\forall\ 1 \le j \le n_\lambda$, $\forall\ \lambda \in \mathfrak{h}^*$ are linearly independent.

Say, $\displaystyle\underbrace{\sum_{\lambda \in \mathfrak{h}^*}}_{\text{finite}} \sum_{j=1}^{n_\lambda} \underbrace{c_{\lambda j}}_{\in \mathbb{C}} m^*_{\lambda j} = 0$ in $F(M) \subseteq M^*$

Evaluate at suitable elements $m_{\lambda i}$ ($\lambda$ occurs in the sum) to show all $c_{\lambda j} = 0$

Claim 2. $m^*_{\lambda j} \in F(M)_\lambda$ i.e. $m^*_{\lambda j}$ are weight vectors

$$(h m^*_{\lambda j})(m_{\mu i}) = m^*_{\lambda j}(i(h) m_{\mu i})$$
$$= m^*_{\lambda j}(h \cdot m_{\mu i})$$

$$= m^*_{\lambda j}(\mu(h)\, m_{\mu i})$$

$$= \mu(h)\, m^*_{\lambda j}(m_{\mu i})$$

$$= \mu(h) \cdot \delta_{\lambda\mu}\, \delta_{ji}$$

$$= \lambda(h)\, \delta_{\lambda\mu}\, \delta_{ji} \quad \left( \because \mu(h)\,\delta_{\lambda\mu} = \lambda(h)\,\delta_{\lambda\mu} \right)$$

$$= \lambda(h)\, m^*_{\lambda j}(m_{\mu i}) \quad \forall \mu, \forall i$$

$$\Rightarrow (h m^*_{\lambda j})(\text{basis of } M) = \lambda(h)\, m^*_{\lambda j}(\text{basis of } M)$$

$$\Rightarrow m^*_{\lambda j} \in F(M)_\lambda \qquad \blacksquare$$

This implies part (I)

(let $m^* \in F(M)_\lambda$, then $m^*\left(\bigoplus_{\mu \neq \lambda} M_\mu\right) = 0$,
so $m^*$ is uniquely determined by $m^*(m_{\lambda i})$ $\forall i$)

(II) (II) is immediate from (I)

as $F(M) = \bigoplus_{\lambda \in \mathfrak{h}^*} (M_\lambda)^*$

(III) $F(F(M)) = \bigoplus_{\lambda \in \mathfrak{h}^*} F(M)^*_\lambda = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^{**}$

Let $F(F(M))_\lambda$ have basis $\left\{ (\tilde{m}_{\mu i})_{i=1}^{n_\mu} : \mu \in \mathfrak{h}^* \right\}$

is a $\mathcal{U}\mathfrak{g}$-module under

$$(X \tilde{m}_{\mu i})(m^*) = \tilde{m}_{\mu i}(i(X) \cdot m^*)$$

$X \in \mathcal{U}\mathfrak{g}$

$$\overset{\text{def}^n}{=} (i(X)\, m^*)(m_{\mu i})$$

$$= m^*(i(i(X))\, m_{\mu i})$$

$$= m^*(X \cdot m_{\mu i}) \quad \forall\, m^* \in M^*$$

So, $\quad m_{\mu i} \longmapsto \tilde{m}_{\mu i}$

i.e, $\displaystyle\sum_{\mu} \sum_{i=1}^{n_\mu} c_{\mu i}\, m_{\mu i} \longmapsto \sum_{\mu} \sum_{i=1}^{n_\mu} c_{\mu i}\, \tilde{m}_{\mu i}$ is a

$\mathfrak{g}$-module isomorphism : $M \longrightarrow F(F(M))$.

## (IV) Recall Basic linear algebra

Say $T : V \to W$ is L.T. b/w f.d. vector spaces over $\mathbb{F}$.

$T^* : W^* \longrightarrow V^*$ is given by

$$T^*(w^*)(v) = w^*(Tv)$$

so given $M, M' \in \mathcal{H}$ and $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, M')$

define $F(\varphi) : F(M') \longrightarrow F(M)$ via

$$F(\varphi)(m'^{*}_{\lambda j})(m_{\mu i}) := m'^{*}_{\lambda j}(\varphi(m_{\mu i}))$$

and extending by linearity ( on )

**claim:-** $F(\varphi)$ is a $\mathfrak{g}$-module map

let $x \in U\mathfrak{g}$. Then

$$F(\varphi)(x m'^{*}_{\lambda j})(\underbrace{m_{\mu i}}_{\text{basis of } M}) = x m'^{*}_{\lambda j}(\varphi(m_{\mu i}))$$

$$= m'^{*}_{\lambda j}(i(x)\,\varphi(m_{\mu i}))$$

$$= m'^{*}_{\lambda j}(\varphi(i(x).m_{\mu i}))$$

$$x(F(\varphi)m'^{*}_{\lambda j})(m_{\mu i}) = F(\varphi)(m'^{*}_{\lambda j})(i(x)m_{\mu i})$$

$$\Rightarrow F(\varphi)(x m'^{*}_{\lambda j})(\underbrace{m_{\mu i}}_{\text{basis of } M}) = x(F(\varphi)m'^{*}_{\lambda j})(\underbrace{m_{\mu i}}_{\text{basis of } M})$$

complete this proof & (IV) (yourself!) $\blacksquare$

(4)(a) **Prop$^n$**: $F: \mathcal{H} \to \mathcal{H}$ is an exact contravariant functor, and $F^2 \equiv id_{\mathcal{H}}$.

"Hint": • Note $\mathcal{H}$ is an abelian category

• $F$ is contravariant by 3(d)-(IV).

$\Rightarrow \underset{\text{on objects}}{F^2 \equiv} id_{\mathcal{H}}$ (by 3(d)-III)

and $F^2$ is covariant on morphisms.

$$(\varphi : M \to N \in \mathcal{H}) \rightsquigarrow (F(\varphi): F(N) \to F(M))$$
$$\downarrow$$
$$\underset{\varphi}{\overset{\cong}{F^2(\varphi)}} : \underset{M}{\overset{\cong}{F^2(M)}} \to \underset{N}{\overset{\cong}{F^2(N)}}$$

e.g. if $\varphi : m_{\mu i} \longmapsto \sum c_{\lambda j} n_{\lambda j}$

then $F^2\varphi : \tilde{m}_{\mu i} \longmapsto \sum c_{\lambda j} \tilde{n}_{\lambda j}$. $\boxtimes$

Exactness of F:-

say $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\eta} M'' \longrightarrow 0$ in $\mathcal{H}$

$\Rightarrow$ • $\varphi : M' \to M$ is 1-1 $\Rightarrow F(\varphi): F(M) \to F(M')$ is onto

• $M \xrightarrow{\eta} M''$ is onto $\Rightarrow F(\eta) : F(M'') \to F(M)$ is 1-1

So $0 \to F(M'') \to F(M) \to F(M') \to 0$

is exact at $F(M'')$ and at $F(M')$.

Claim: It is also exact at $F(M)$.

(Do yourself!) $\blacksquare$

4(b) Now come to $F$ acting on $\mathcal{O}$.

Cor 1. $F(L(\lambda)) \cong L(\lambda)$.

proof (1) $F(L(\lambda)) \neq$ simple

$\Rightarrow$ it has non-zero proper sub-module

$\Rightarrow F^2(L(\lambda))$ has non-zero proper quotient

$\Rightarrow L(\lambda)$ is not simple. $\blacksquare$

proof (2) $F(L(\lambda))$ has a highest weight vector of wt $\lambda$,
so $F(L(\lambda)) \twoheadrightarrow L(\lambda)$ and $\operatorname{char} F(L(\lambda)) = \operatorname{char} L(\lambda)$ $\blacksquare$

4(c)

Cor 2. Say $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_k = M$ is
a chain of objects in $\mathcal{O}$. Then

$F(M) \in \mathcal{O}$ and has a chain of submodules with the subquotients $F(M_{j+1}/M_j)$ occurring in the reverse order.

To prove $F(M) \in \mathcal{O}$ use exact/contravariance of $F$ and induction on $\ell(M)$ (and $F(L(\lambda)) = L(\lambda)$)

4(d)
Cor 3. By letting the above chain be maximal:
$$\ell(F(M)) = \ell(M) \quad \forall M \in \mathcal{O}.$$

(5) proof of thm (2):-

5(a) Propn: Let $\lambda, \mu \in \mathfrak{h}^*$ and $\mathrm{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) \neq 0$

i.e., $0 \longrightarrow L(\lambda) \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0$

e.g. $0 \subseteq \dots \subseteq N \subseteq N_{max} \subseteq M(\mu) \Rightarrow 0 \subseteq \frac{N_{max}}{N} \subseteq \frac{M(\mu)}{N}$

$\ni \quad \frac{M(\mu)}{N} \cong L(\mu) \quad \& \quad \frac{N_{max}}{N} \cong L(\lambda)$

so $L(\mu)$ & $L(\lambda)$ can't occur as direct sum in $M$ as otherwise $L(\lambda)$ also occurs in $M(\mu)/N$.

Then either:

(a) $\mu \underset{\neq}{\geq} \lambda$ and

$L(\mu), L(\lambda)$ are the two top J-H factors in $M(\mu)$

say $0 \longrightarrow L(\lambda) \longrightarrow M(\mu)/N \longrightarrow L(\mu) \longrightarrow 0$;

or

(b) $\mu \underset{\neq}{\leq} \lambda$

and $0 \longrightarrow \underset{F(L(\lambda))}{\overset{\shortparallel}{L(\lambda)}} \longrightarrow F\left(M(\lambda)/N\right) \longrightarrow \underset{F(L(\mu))}{\overset{\shortparallel}{L(\mu)}} \longrightarrow 0$

(In particular, $\chi_\lambda = \chi_\mu$).

Moreover, $\dim \operatorname{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) < \infty$. ☒

## 5(b) observation:-

Say $\cdots \longrightarrow A \xrightarrow{d} B \xrightarrow{d'} C \longrightarrow \cdots\cdots$ is a sequence of vector spaces exact at $B$.

If $A, C$ are f.d., so is $B$.

Proof: $\dim(B) = \dim(\operatorname{im}(d')) + \dim(\ker d')$
$\leq \dim C + \dim(\operatorname{im} d)$

$\leq \dim C + \dim A < \infty$ $\blacksquare$

5(c) Proof of thm (2) $\rightarrow$ (i.e. $\mathrm{Ext}^1_{\mathcal{O}}(M, M')$ is f.d. $\forall M, M' \in \mathcal{O}$)

we first show this for $M = L(\lambda)$ by induction on $\ell(M') \geq 0$.

- $\ell = 0$ holds trivially
- $\ell = 1$ holds by 5(a).
- induction step: say $0 \to M_{\ell-1} \to M' \to L(\mu) \to 0$

Applying $\mathrm{Hom}_{\mathcal{O}}(L(\lambda), -)$; we get long exact sequence

$$0 \to \mathrm{Hom}(L(\lambda), M_{\ell-1}) \to \mathrm{Hom}(L(\lambda), M') \to \mathrm{Hom}(L(\lambda), L(\mu))$$
$$\to \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), M_{\ell-1}) \to \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), M') \to \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\mu))$$

dim $< \infty$ by induction

dim $< \infty$ by 5(a)

$\to \cdots$

so, $\dim(\mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), M')) < \infty$ by 5(b).

Now prove the general case by applying $\mathrm{Hom}_{\mathcal{O}}(-, M')$ to all $M \in \mathcal{O}$. $\blacksquare$

# (6) Projective Modules in $\mathcal{O}$

(a) **Definition**: An object $P$ in an abelian category $\mathcal{A}$ is projective if $\forall$ epimorphism $N \overset{f}{\twoheadrightarrow} M$ and a map $g: P \longrightarrow M$ in $\mathcal{A}$ $\exists\ h: P \longrightarrow N$ making the diagram

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

commute.

6(b) Reverse arrows; change epimorphism to monomorphism, defines injective objects in $\mathcal{A}$.

6(C) We say $\mathcal{A}$ has enough projectives if $\forall\ M \in \mathcal{A}$ $\exists$ projective $P$ and an epimorphism $: P \twoheadrightarrow M$ in $\mathcal{A}$.

6(d) Goal: To prove the following thm.

Thm (I) $\mathcal{O}_\theta$ has enough projectives $\forall\ \theta \in \mathfrak{h}^*/(W_j, \cdot)$

(II) Hence $\mathcal{O}$ has enough projectives.

(III) Every projective has a (finite) filtration whose subquotients are Verma-modules.

## (7) Basics on projective-modules:-

(a) Prop$^n$ TFAE in $\mathcal{A}$ for an object $P$:

(I) $P$ is projective

(II) $\mathrm{Ext}^1_{\mathcal{A}}(P, M) = 0$ for all objects $M$ in $\mathcal{A}$

(III) $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

7(b) Cor. say $P = P' \oplus P''$ in $\mathcal{A}$

Then $P$ is projective $\Leftrightarrow$ $P', P''$ are projectives.

## (8)(a) Proof of th$^m$ 6(d) $(1) \Rightarrow (2)$

Say $M \in \mathcal{O}$ $\Rightarrow$ $M = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\cdot)} M_\theta$ ← only finitely many $M_\theta \neq 0$

say $M_{\theta_1}, M_{\theta_2}, \dots, M_{\theta_n}$ are only non-zero.

By part (1), $\exists$ projectives $P_1, \dots, P_n$ ; $P_j \in M_{\theta_j}$

s.t. $P_j \twoheadrightarrow M_{\theta_j} \quad \forall j$

$$\Rightarrow \bigoplus_{j=1}^{n} P_j =: P \text{ surjects onto } \bigoplus_{j=1}^{n} M_{\theta_j} = M$$

By 7(b), it is enough to show each $P_j$ is projective in $\mathcal{O}$.

Given $N \in \mathcal{O}$ ; write $N = N_{\theta_j} \oplus N_{\theta_1'} \oplus \cdots \oplus N_{\theta_s'}$

Now $\quad \operatorname{Ext}^1_{\mathcal{O}}(P_j, N_{\theta_j}) = 0$ $\quad$ [$\because$ $P_j$ is projective in $\mathcal{O}_{\theta_j}$ by Prop 7(a) (II)]

and $\operatorname{Ext}^1_{\mathcal{O}}(\underset{\mathcal{O}_{\theta_j}}{\underset{\cap}{P_j}}, \underset{\mathcal{O}_{\theta_l'}}{\underset{\cap}{N_{\theta_l'}}}) = 0$ $\quad$ [$\because$ $\mathcal{O}_{\theta_j} \neq \mathcal{O}_{\theta_l'}$]

$\Rightarrow$ using the long exact sequence of $\operatorname{Ext}_{\mathcal{O}}(P_j, -)$, inductively, we get $\quad$ ( $\operatorname{Ext}^0_{\mathcal{O}}(P_j, -) = \operatorname{Hom}_{\mathcal{O}}(P_j, -)$)

$$\operatorname{Ext}^1_{\mathcal{O}}(P_j, N) = 0 \quad \forall N \in \mathcal{O}$$

By Prop 7(a) (II), $P_j$ is projective in $\mathcal{O}$

By Cor 7(b), $\bigoplus_j P_j$ is projective in $\mathcal{O}$. ▨

8(b) Examples of a projective in $\mathcal{O}_\theta$ over semi-simple $\mathfrak{g}$.

[Note if $\lambda = \sqrt{3}\alpha$ then $\underset{-\sqrt{3}\alpha}{\underset{\shortparallel}{s_\alpha\lambda}} = \lambda - 2\sqrt{3}\alpha$

then $s_\alpha\lambda$ & $\lambda$ are not comparable.
i.e. $s_\alpha\lambda \not\leq \lambda$.]

**Lemma**: Say $\mu \in W\cdot\lambda = \theta$ is maximal in partial order $\leq$.
Then $M(\mu)$ is projective in $\mathcal{O}_\theta$ and hence in $\mathcal{O}$ (by 7(a))

proof: say,

$$\begin{array}{ccc} & & N \\ & & \big\downarrow f \\ M(\mu) & \xrightarrow[g]{} & M \end{array} \quad \text{in } \mathcal{O}_\theta$$

- If $g(M(\mu)) = 0$ then let $h : M(\mu) \longrightarrow 0_N$
- else $g(\underbrace{m_\mu}_{\text{highest weight vector}}) \neq 0$ in $M_\mu$, and $M$ is highest weight module.

choose $n_\mu \in f^{-1}(g(m_\mu))$

- If $\eta^+ . n_\mu = 0$ then $n_\mu$ is maximal in $N \in \mathcal{O}_\theta$ so by universal property of Verma module $M(\mu)$,

the map $h : \underset{\in U(\eta^-)}{X} m_\mu \longmapsto X n_\mu$ would work.

$$\begin{tikzcd} X m_\mu \arrow[r, mapsto, "h"] \arrow[dr, "g"'] & X n_\mu \arrow[d, "f"] \\ & X f(n_\mu) = X g(m_\mu) \end{tikzcd}$$

- If $e_i . n_\mu \neq 0$ ; $e_i . n_\mu \in N_{\mu+\alpha_i}$

But $\mu$ is maximal in $\leq$, and $N \in \mathcal{O}_\theta$

$\Rightarrow N_{\mu+\alpha} = 0$

which is contradiction

$\therefore e_i . n_\mu = 0. \ \forall i$

$\Rightarrow \eta^+ . n_\mu = 0$ $\boxtimes$

## 8(C) Example of a projective in $\mathcal{O}_{\{0,-\alpha\}}$ over $\mathfrak{sl}_2(\mathbb{C})$.

- $\lambda = 0$ is maximal ; $\lambda = -\alpha$

$\Rightarrow M(0)$ is projective, ,

$M(0) \twoheadrightarrow L(0)$. Now, we claim:

$\dfrac{U\mathfrak{g}}{U\mathfrak{g}\cdot e^2 + U\mathfrak{g}(h+\lambda(h))} = M_{2,-\alpha} = \text{projective} \twoheadrightarrow L(-2) = L(-\alpha)$

<u>Note:</u> $0 \to \underset{\underset{M(0)}{\parallel}}{M(\lambda+\alpha)} \to M_{2,-\alpha} \to \underset{\underset{M(-\alpha)}{\parallel}}{M(\lambda)} \to 0$

$\because$ $M(0)$ & $M(-\alpha) \in \mathcal{O}_{\theta_0}$

so $M_{2,-\alpha} \in \mathcal{O}_{\theta_0}$

To show $M_{2,-\alpha}$ is projective in $\mathcal{O}$, it is enough to show $\mathrm{Hom}_{\mathcal{O}_{\theta_0}}(M_{2,-\alpha}, -)$ is exact.

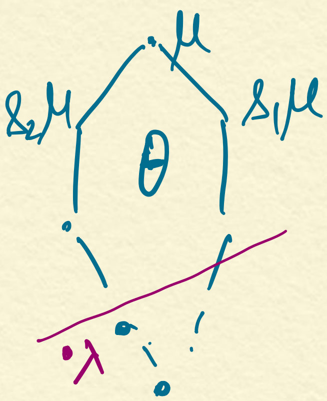
Why? More generally, we have:

<u>§(d)</u> Back to $\mathfrak{g}$ = semi-simple$|_{\mathbb{C}}$

given $\lambda \in \mathfrak{h}^*$ and $\theta \in \mathfrak{h}^*/(W,\cdot)$, define

- $\ell_{\lambda,\theta} \in \mathbb{Z}_{>0}$ to be $1 + \max\left\{ ht(\mu) ;\ \begin{matrix}\mu\in\theta \\ \mu \geq \lambda\end{matrix} \right\}$

so $(\lambda + \alpha_{i_1} + \cdot + \alpha_{i_\ell} \neq \text{any } \mu \in \theta)$



- $\mathcal{O}_{\lambda,\theta} := \{ M \in \mathcal{O} : (\eta^+)^{\ell_{\lambda,\theta}} . M_\lambda = 0 \}$

full sub-category

- $P_{\lambda,\theta} := \mathcal{U}\mathfrak{g} \cdot 1 \Big/ \Big( \sum_{i \in I} \mathcal{U}\mathfrak{g}(h_i - \lambda(h_i)) + \mathcal{U}\mathfrak{g} \cdot (\eta^+)^{\ell_{\lambda,\theta}} \cdot 1 \Big)$

(e.g. $M_{2,-\alpha} = \mathcal{U}\mathfrak{g}.1 \big/ \big(\mathcal{U}\mathfrak{g} \cdot \underset{(\eta^+)^2}{\overset{\shortparallel}{e^2}} + \mathcal{U}\mathfrak{g} \cdot (h - \lambda(h))\big)$ is example of $P_{\lambda,\theta}$)

So $P_{\lambda,\theta} \in \underbrace{\mathcal{O}_{\lambda,\theta}}$

abelian subcategory of $\mathcal{O}$.

<u>Prop$^n$</u> (I) $P_{\lambda,\theta}$ is projective in $\mathcal{O}_{\lambda,\theta}$, as $\mathrm{Hom}_{\mathcal{O}_{\lambda,\theta}}(P_{\lambda,\theta}, M) \cong M_\lambda$ ← exact

(II) $P_{\lambda,\theta}$ has a $\underbrace{\text{Verma flag}}$

(filtration with subquotients are Verma modules)

(Used to prove part (III) of th$^m$ 6(d))

<u>Example</u> over $\mathfrak{sl}_3(\mathbb{C})$

quotient by $(\eta^+)^2$.

$\underbrace{\mathcal{U}(\eta^+) / \mathcal{U}(\eta^+)(\eta^+)^2} \cong \mathbb{C} \oplus \mathbb{C}e_{\alpha_1} \oplus \mathbb{C}e_{\alpha_2} \oplus \mathbb{C}e_{\alpha_1+\alpha_2}$

would" get a filtration

$$0 \subseteq M(\lambda+\alpha_1+\alpha_2) \subseteq M_2 \subseteq M_3 \subseteq M_4$$

$$M_2 = M(\lambda+\alpha_1), \quad M_3 = M(\lambda+\alpha_2), \quad M_4 = M(\lambda)$$

This is the starting point in constructing projections in $\mathcal{O}_\theta$.

## 9 connecting $\mathcal{O}_\theta$ to a finite-dim algebra

Thm 9(a) Each $\mathcal{O}_\theta$ is equivalent to the abelian category of f.d. right modules over a f.d. associative $\mathbb{C}$-algebra $B_\theta$.

## 9(b) How to construct $B_\theta$?

Thm (Krull-Schmidt) If an $R$-module $M$, satisfies a.c.c. and d.c.c ($\overset{\text{J-H}}{\Longleftrightarrow}$ finite length) then one has $M = M_1 \oplus \dots \oplus M_n$, and all $M_j \neq 0$ are unique up to reordering and isomorphism. ($M_j$'s are indecomposable.)

9(C) 9(b) $\Rightarrow \left\{ \begin{array}{l} \text{Indecomposable} \\ \text{projectives in } \mathcal{O}_\theta \end{array} \right\} =: \mathcal{P}_\theta$

**Th^m** (I) $|\mathcal{P}_\theta| = |\theta|$, say $\theta = \{\lambda_1, \dots, \lambda_s\}$

(II) Each $L(\lambda_j)$ is associated with a unique indecomposable projective $P_j \in \mathcal{P}_\theta$

with

$$\begin{tikzcd} & M(\lambda_j) \arrow[d, two heads] \\ P_j \arrow[ur] \arrow[r, two heads] & L(\lambda_j) \end{tikzcd}$$

(III) Answer to 9(b) (i.e. How to construct $B_\theta$?)

let $Q := \bigoplus_{j=1}^{s} P_j$

= projective in $\mathcal{O}_\theta$

$$\Rightarrow B_\theta = \mathrm{End}_{\mathcal{O}_\theta}(Q) = \underbrace{H_{\mathcal{O}_\theta}(Q, Q)}_{f.d.}$$

(IV) The functor $\mathrm{Hom}_{\mathcal{O}_\theta}(Q, -) : \mathcal{O}_\theta \to (\text{mod-}B_\theta)$ is an equivalence of (abelian) categories, with inverse equivalence $Q \otimes_{B_\theta} -$.

$\boxtimes$