# SIX LECTURES ON BERNSTEIN–GELFAND–GELFAND CATEGORY $\mathcal{O}$

## WEEK 3 OF ONLINE ADVANCED INSTRUCTIONAL SCHOOL DECEMBER 16–21, 2024

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Throughout, $\mathfrak{g}$ denotes a (finite-dimensional) complex semisimple Lie algebra, with a fixed Cartan subalgebra $\mathfrak{h}$, root system $\Phi$, base of simple roots $\Delta = \{\alpha_i : i \in I\}$, and Chevalley generators $\{e_i, f_i, h_i = [e_i, f_i] : i \in I\}$.

The **structure** of this series of six lectures is as follows:

(1) Recap of basics of complex semisimple Lie algebras (the above, plus fundamental weights); Verma modules $M(\lambda)$ and their quotients; the center $Z(U(\mathfrak{g}))$; the Harish-Chandra projection $\mathrm{pr}(\cdot) : Z(U(\mathfrak{g})) \to U(\mathfrak{h})$; the center acting on a highest weight module $M(\lambda) \twoheadrightarrow V$ via the central character $\chi_\lambda = \lambda \circ \mathrm{pr}$.

(2) More basics of highest weight modules (locally $\mathfrak{n}^+$-finite); definition of the BGG category $\mathcal{O}$; examples of objects – including the following object over $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$:
$$M_{\ell,\lambda} := U(\mathfrak{g}) \cdot 1 / \left[ U(\mathfrak{g}) \cdot e^\ell + U(\mathfrak{g}) \cdot (h - \lambda(h)) \right].$$
Basic properties of $\mathcal{O}$ (weights in finitely many cones; finite filtration via highest weight modules – e.g. for $M_{\ell,\lambda}$).

(3) Simple quotients $L(\lambda)$ of highest weight modules; classification of simple objects in $\mathcal{O}$; ascending/descending chains of submodules and the Jordan–Hölder theorem.
**Theorem:** *All objects in $\mathcal{O}$ have finite length.*
*Corollary:* $\mathcal{O}$ is closed under taking sub-objects (and hence, an abelian category).

(4) (Formal) Characters of objects in $\mathcal{O}$ (e.g. $M(\lambda) \twoheadrightarrow V$ and $M_{\ell,\lambda}$); the Grothendieck group $K(\mathcal{O})$; linear independence of Verma module characters.
**Theorem:** $K(\mathcal{O}) \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[L(\lambda)] \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \mathbb{Z}[M(\lambda)]$, *and* $\mathrm{char}(\cdot)$ *is 1-1 on* $K(\mathcal{O})$.
More strongly: given a twisted Weyl group orbit $\theta = \theta_\lambda = W \bullet \lambda \in \mathfrak{h}^*/(W, \bullet)$, define $K(\mathcal{O})_\theta := \bigoplus_{w \in W} \mathbb{Z}[L(w \bullet \lambda)]$. Then $K(\mathcal{O})_\theta := \bigoplus_{w \in W} \mathbb{Z}[M(w \bullet \lambda)]$.
This involves showing that upon suitably (re)ordering the elements of a block $\theta = \theta_\lambda = W \bullet \lambda$, the decomposition matrix of Jordan–Hölder multiplicities of simple modules in Verma modules is uni-triangular – i.e., (upper) triangular with diagonal entries 1.

(5) Vanishing of Hom's and Ext's between highest weight modules with differing central characters; Hom-spaces are finite-dimensional in $\mathcal{O}$;

defining the block $\mathcal{O}_\theta \subseteq \mathcal{O}$ as the full subcategory of those modules $M$ for which all Jordan–Hölder factors $L(\lambda)$ have $\chi_\lambda = \chi_\theta$ (i.e., $\lambda \in \theta$);

Example: block decomposition of the modules $M_{2,\lambda}$ over $\mathfrak{sl}_2(\mathbb{C})$.

**Theorem:** ***(1)*** *$\mathcal{O}_\theta = \{M \in \mathcal{O} : \forall z \in Z(U(\mathfrak{g}))\, \exists k \geq 0$ with $(z - \chi_\theta(z))^k \cdot M = 0\}$.*
*(And in this case, $k = \ell(M)$ works for all $z$, and for all $M$.)*
*Moreover, each $\mathcal{O}_\theta$ is an abelian subcategory of $\mathcal{O}$.*

***(2)*** $\mathcal{O} = \bigoplus_{\theta \in \mathfrak{h}^*/(W,\bullet)} \mathcal{O}_\theta$.

***(3)*** *All Hom's and* Ext*'s between objects in unequal blocks vanish.*
(After mentioning the long exact sequences of Ext's.)

(6) The "restricted duality" functor on the Harish-Chandra category $\mathcal{H}$: exact, contravariant, fixes each $L(\lambda)$, reverses short exact sequences, and preserves $\mathcal{O}$.

**Theorem:** $\operatorname{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) \neq 0$ *if and only if either* ***(a)*** *$\mu > \lambda$ and $L(\mu), L(\lambda)$ are the top two terms in a Jordan–Hölder series for the Verma module $M(\mu)$ – say $0 \to L(\lambda) \to M(\mu)/M_{-2} \to L(\mu)$; or* ***(b)*** *$\mu < \lambda$ and the extension is the restricted dual of the above short exact sequence (with $\mu, \lambda$ switched).*
*Moreover,* $\operatorname{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda))$ *is then finite-dimensional.*

*Application:* $\operatorname{Ext}^1_{\mathcal{O}}(M, M')$ is finite-dimensional for all $M, M' \in \mathcal{O}$.

Projective objects in $\mathcal{O}$ – e.g. $M_{2,-\alpha}$ over $\mathfrak{sl}_2(\mathbb{C})$, and $M(\lambda)$ over any semisimple $\mathfrak{g}$, provided $\lambda$ is maximal in its twisted $W$-orbit.

We also state the following results (part (3) was not stated in the lectures):

**Theorem:** ***(1)*** *$\mathcal{O}_\theta$ and hence $\mathcal{O}$ has enough projectives and injectives.*

***(2)*** *Let $P(\lambda) \twoheadrightarrow M(\lambda) \twoheadrightarrow L(\lambda)$ be the (indecomposable) projective cover of $L(\lambda)$ in $\mathcal{O}$. (Thus $P(\lambda) \in \mathcal{O}_{\theta_\lambda}$.) Then $P(\lambda)$ has a finite filtration whose subquotients are Verma modules $M(w \bullet \lambda)$. This last property is preserved under taking direct summands.*

***(3)*** *In each block $\mathcal{O}_\theta$ and hence in $\mathcal{O}$, one has BGG reciprocity: the projective-to-Verma multiplicity $[P(\lambda) : M(\mu)]$ equals $[M(\mu) : L(\lambda)]$ for all weights $\lambda, \mu \in \mathfrak{h}^*$.*

***(4)*** *Finally, we link to the other modules of this AIS: given any block $\theta \in \mathfrak{h}^*/(W, \bullet)$, consisting of weights $\lambda_1, \dots, \lambda_s$, let $n_j \in \mathbb{Z}_{>0}$ for all $1 \leq j \leq s$, and define $Q := \bigoplus_{j=1}^s P(\lambda_j)^{\oplus n_j}$. Then $Q$ is a projective generator for $\mathcal{O}_\theta$, i.e. the Hom-space from $Q$ to any object of the block is nonzero.*

***(5)*** *The space $B_\theta :=$ $\operatorname{End}_{\mathcal{O}}(Q, Q)$ is a finite-dimensional associative $\mathbb{C}$-algebra, whose Morita equivalence class is independent of the integers $n_j > 0$. Moreover, the functor $Hom_{\mathcal{O}}(Q, -)$ is an equivalence from $\mathcal{O}_\theta$ to the category of finite dimensional right $B_\theta$-modules, with inverse equivalence functor $Q \otimes_{B_\theta} -$.*

The notes for these lectures are all scribed by Shushma Rani and can be found elsewhere. Below can be found the notes of the first two Tutorial sessions, and some results from the sixth lecture; the rest of the Tutorial sessions were also scribed by Shushma.

# 1. Day 1: 16 Dec, 2024 – Basics of (highest) weight modules – Tutor: G.V. Krishna Teja

We record a fact that is very useful in today's tutorial: by the PBW theorem, the multiplication map is a $\mathbb{C}$-vector space isomorphism:

$$mult : U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \to U(\mathfrak{g}). \tag{1.1}$$

**Question 1.** Show that every Verma module is isomorphic to $U(\mathfrak{n}^-)$. More precisely, the map : $X \cdot 1 \mapsto X \cdot m_\lambda$ is an isomorphism of free $U(\mathfrak{n}^-)$ modules of rank one.

*Sketch of proof.* This is an exercise in using the PBW theorem. For $\lambda \in \mathfrak{h}^*$, the Verma module is

$$M(\lambda) := \frac{U(\mathfrak{g})}{\sum_{i\in I} U(\mathfrak{g})e_i + \sum_{i\in I} U(\mathfrak{g})(h_i - \lambda(h_i))}.$$

Recall that $\mathfrak{n}^+$ is generated as a Lie algebra by the $e_i$ (via taking iterated commutator brackets in the associative algebra $U(\mathfrak{g})$). Thus,

$$\sum_{i\in I} U(\mathfrak{g})e_i \supseteq U(\mathfrak{g})\mathfrak{n}^+.$$

The reverse inclusion is immediate because $e_i \in \mathfrak{n}^+$ for all $i$. Thus, $\sum_{i\in I} U(\mathfrak{g})e_i = U(\mathfrak{g})\mathfrak{n}^+$.

Next, let $\mathfrak{m}_\lambda$ be the maximal "evaluation ideal" of the commutative $\mathbb{C}$-algebra $U(\mathfrak{h})$ that is generated by all $h_i - \lambda(h_i)$. Using (1.1), one shows that

$$\sum_{i\in I} U(\mathfrak{g})(h_i - \lambda(h_i)) = U(\mathfrak{n}^-) \cdot U(\mathfrak{h}) \cdot \sum_{i\in I} U(\mathfrak{n}^+)(h_i - \lambda(h_i))$$

under multiplication. Now using that $U(\mathfrak{n}^+) = \mathbb{C} \oplus U(\mathfrak{n}^+)\mathfrak{n}^+$, we see that each summand above on the right is contained in $\mathbb{C} \cdot \mathfrak{m}_\lambda \oplus U(\mathfrak{h}) \cdot U(\mathfrak{n}^+)\mathfrak{n}^+$. Hence the "Verma denominator"

$$\sum_{i\in I} U(\mathfrak{g})e_i + \sum_{i\in I} U(\mathfrak{g})(h_i - \lambda(h_i)) = U(\mathfrak{g})\mathfrak{n}^+ \oplus U(\mathfrak{n}^-)\mathfrak{m}_\lambda. \tag{1.2}$$

Similarly, by (1.1) the "Verma numerator" is

$$\begin{aligned} U(\mathfrak{n}^-) \cdot U(\mathfrak{h}) \cdot U(\mathfrak{n}^+) &= \left[U(\mathfrak{n}^-) \cdot U(\mathfrak{h}) \cdot U(\mathfrak{n}^+)\mathfrak{n}^+\right] \oplus U(\mathfrak{n}^-) \cdot U(\mathfrak{h}) \\ &= \left[U(\mathfrak{g})\mathfrak{n}^+\right] \oplus U(\mathfrak{n}^-) \cdot (\mathfrak{m}_\lambda \oplus \mathbb{C}) \\ &= \left[U(\mathfrak{g})\mathfrak{n}^+ \oplus U(\mathfrak{n}^-)\mathfrak{m}_\lambda\right] \oplus U(\mathfrak{n}^-). \end{aligned}$$

Now taking the quotient, one can show the result. □

**Question 2.** If $M$ is an $\mathfrak{h}$-semisimple $\mathfrak{g}$-module, then so is every submodule $N$ of $M$.

*Proof.* We provide two approaches. The first is motivated by the $\mathfrak{sl}_2(\mathbb{C})$-case, where $M$ is merely $h$-semisimple. Suppose $0 \neq N \leq M$ is a submodule, and $n = n_1 + \cdots + n_k \in N$, with

$n_j \in M_{\mu_j}$ for pairwise distinct $h$-eigenvalues $\mu_j = \mu_j(h)$. Applying increasing powers of $h$,

$$\begin{aligned} h^0 \cdot (n_1 + \cdots + n_k) &= \sum_{j=1}^k n_j \in N, \\ h^1 \cdot (n_1 + \cdots + n_k) &= \sum_{j=1}^k \mu_j(h) n_j \in N, \\ &\vdots \\ h^{k-1} \cdot (n_1 + \cdots + n_k) &= \sum_{j=1}^k \mu_j(h)^{k-1} n_j \in N. \end{aligned}$$

In other words, using compact notation,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1(h) & \mu_2(h) & \cdots & \mu_k(h) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1(h)^{k-1} & \mu_2(h)^{k-1} & \cdots & \mu_k(h)^{k-1} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} \in \begin{pmatrix} N \\ N \\ \vdots \\ N \end{pmatrix}.$$

Denoting the Vandermonde matrix on the left by $A$, we note that $A$ is invertible, so

$$A^{-1} A \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} \in \begin{pmatrix} N \\ N \\ \vdots \\ N \end{pmatrix}.$$

Therefore $n_j \in N$, as claimed.

The proof is similar for general $\mathfrak{g}$. Let $n = n_1 + \cdots + n_k \in N$, with $n_j \in M_{\mu_j}$ for pairwise distinct weights $\mu_j \in \mathfrak{h}^*$. For each $1 \leq i < j \leq k$, the nonzero linear functional $\mu_i - \mu_j$ vanishes on a (codimension one) hyperplane $H_{ij} \subseteq \mathfrak{h}$. But as $\mathbb{C}$ is an infinite field, $\bigcup_{i<j} H_{ij}$ is not all of $\mathfrak{h}$. Choose any $h$ in the complement; now $\mu_j(h)$ are distinct scalars for all $j$, and we can repeat the above proof for $\mathfrak{sl}_2(\mathbb{C})$ to arrive at the same conclusion for arbitrary $\mathfrak{g}$.

The second approach is to work with induction on $k \geq 1$. The result is trivial for $k = 1$; else since $\mu_{k-1} \neq \mu_k$, there exists $h \in \mathfrak{h}$ such that $\mu_{k-1}(h) \neq \mu_k(h)$. Now apply the scalar $\mu_{k-1}(h)$ and the operator $h$ to the above assertion $n_1 + \cdots + n_k \in N$. As $N$ is stable under both applications, we have:

$$\begin{aligned} \mu_{k-1}(h) v_1 + \cdots + \mu_{k-1}(h) v_{k-1} + \mu_{k-1}(h) v_k &\in N, \\ \mu_1(h) v_1 + \cdots + \mu_{k-1}(h) v_{k-1} + \mu_k(h) v_k &\in N. \end{aligned}$$

Subtracting the first from the second, we have:

$$\sum_{j=1}^{k-2} (\mu_j(h) - \mu_{k-1}(h)) v_j + (\mu_k(h) - \mu_{k-1}(h)) v_k \in N.$$

As there are fewer than $k$ terms here, by the induction hypothesis each summand is in $N$; hence by choice of $h$, $v_k \in N$. Now subtract from the original condition $v_1 + \cdots + v_k \in N$ and apply the induction hypothesis once again to obtain: all $v_j \in N$. $\square$

**Question 3.** Show that the Harish-Chandra projection $\mathrm{pr}(\cdot)$, and hence every central character $\chi_\lambda(\cdot)$ for $\lambda \in \mathfrak{h}^*$, is a $\mathbb{C}$-algebra map.

*Proof.* It suffices to show that $\mathrm{pr}(\cdot) : Z(U(\mathfrak{g})) \to U(\mathfrak{h})$ is an algebra map, since so is each $\lambda(\cdot) : U(\mathfrak{h}) \to \mathbb{C}$. First fix an ordering of the positive roots, say $\beta_1, \dots, \beta_N \in \Phi^+$; now via (1.1), each $X \in U(\mathfrak{g})$ is a finite linear combination of words $F^K H^J E^{K'}$, where

$$F^K := f_{\beta_1}^{k_1} \cdots f_{\beta_N}^{k_N}, \qquad H^J := \prod_{i \in I} h_i^{j_i}, \qquad E^{K'} := e_{\beta_1}^{k'_1} \cdots e_{\beta_N}^{k'_N}.$$

Now recall from Lecture 1 (of this week): every central element $z \in Z(U(\mathfrak{g}))$ is of the form

$$z = \sum_{K, K' \in \mathbb{Z}_{\geq 0}^N,\, J \in \mathbb{Z}_{\geq 0}^I} c_{K,J,K'} F^K H^J E^{K'}, \quad \text{with} \quad \sum_{j=1}^N k_j \beta_j = \sum_{j=1}^N k'_j \beta_N$$

and all $c_{K,J,K'} \in \mathbb{C}$. In particular, we saw that

$$K = \mathbf{0} \Longleftrightarrow K' = \mathbf{0}. \tag{1.3}$$

Now let

$$z = \sum_{K, K' \in \mathbb{Z}_{\geq 0}^N,\, J \in \mathbb{Z}_{\geq 0}^I} c_{K,J,K'} F^K H^J E^{K'}, \quad y = \sum_{K_1, K'_1 \in \mathbb{Z}_{\geq 0}^N,\, J_1 \in \mathbb{Z}_{\geq 0}^I} d_{K_1,J_1,K'_1} F^{K_1} H^{J_1} E^{K'_1}$$

be central. By (1.3), we have

$$\mathrm{pr}(z)\,\mathrm{pr}(y) = \sum_J c_{\mathbf{0},J,\mathbf{0}} H^J \cdot \sum_{J_1} d_{\mathbf{0},J_1,\mathbf{0}} H^{J_1}.$$

Next, a typical monomial in $zy$ is

$$F^K H^J E^{K'} \cdot F^{K_1} H^{J_1} E^{K'_1}. \tag{1.4}$$

Now if $K'_1 \neq \mathbf{0}$ then $E^{K'_1} \in U(\mathfrak{n}^+)\mathfrak{n}^+ \subseteq U(\mathfrak{g})\mathfrak{n}^+$. Now writing the monomial (1.4) in a PBW basis obtained from (1.1) will never yield a component in $U(\mathfrak{h})$. So $\mathrm{pr}(\cdot)$ sends (1.4) to zero.

Similarly, if $K \neq \mathbf{0}$ then the monomial (1.4) is in $\mathfrak{n}^- U(\mathfrak{n}^-)$, hence is killed by $\mathrm{pr}(\cdot)$.

Thus, the only monomials (1.4) which contribute to $\mathrm{pr}(zy)$ satisfy: $K = K'_1 = 0$. But then by (1.3), $K' = K_1 = 0$. So,

$$\mathrm{pr}(zy) = \mathrm{pr}\left( \sum_{J, J_1} c_{\mathbf{0},J,\mathbf{0}} d_{\mathbf{0},J_1,\mathbf{0}} H^{J+J_1} \right),$$

and this equals $\mathrm{pr}(z)\,\mathrm{pr}(y)$ as above. $\square$

**Remark 1.1.** The implicit point in the final question above is that the Harish-Chandra projection seems to depend on the choice of PBW basis for $\mathfrak{n}^\pm$ and $\mathfrak{h}$. But actually, it is defined in a "coordinate-free" manner as follows: write

$$U(\mathfrak{g}) = (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+) \oplus U(\mathfrak{h}).$$

Then $\mathrm{pr}(\cdot)$ is simply the projection map to the second factor of $U(\mathfrak{h})$, and arguments similar to the ones above help show that it is an algebra map when restricted to $Z(U(\mathfrak{g}))$.

## 2. Day 2: 17 Dec, 2024 – $U(L)$ is Noetherian and without zerodivisors – Tutor: G.V. Krishna Teja

The overarching goal is to show:

**Question 4.** If $L$ is a finite-dimensional Lie algebra over any field $\mathbb{F}$, prove that $U(L)$ is a Noetherian $\mathbb{F}$-algebra without zerodivisors. Moreover, prove that every submodule of a finitely generated $L$-module is finitely generated.

If this holds, then as a corollary for $\mathbb{F} = \mathbb{C}$ and $L$ a semisimple Lie algebra, we have that the BGG Category $\mathcal{O}$ is closed under taking submodules.

We answer/prove the assertions in the question systematically, beginning with the "no zerodivisors" assertion.

*Proof that $U(L)$ has no zerodivisors.* Recall that $U(L)$ is an $\mathbb{F}$-algebra with a filtration

$$0 = F^{-1}(U(L)) \subseteq \mathbb{F} = F^0(U(L)) \subseteq \cdots \subseteq F^n(U(L)) \subseteq \cdots$$

We also recall that the associated graded algebra of a filtered algebra (in this case $U(L)$) is simply

$$gr_F U(L) := \bigoplus_{n \geq 0} F^n(U(L))/F^{n-1}(U(L)),$$

with the multiplication defined as follows. If

$$a \in F^n(U(L)) \setminus F^{n-1}(U(L)), \qquad b \in F^m(U(L)) \setminus F^{m-1}(U(L))$$

for some $n, m \in \mathbb{Z}_{\geq 0}$, then firstly we define $gr_F(a)$ to be the image of $a$ in $F^n(U(L))/F^{n-1}(U(L)) \subseteq gr_F(U(L))$; we also define $gr_F(0) := 0$; and secondly we define the product on $gr_F(U(L))$ to be $gr_F(a) \cdot gr_F(b) := gr_F(ab)$, where $ab \in F^{n+m}(U(L))$.

Returning to the question, $U(L)$ is a filtered $\mathbb{F}$-algebra whose associated graded algebra is $\mathrm{Sym}(L)$, by the PBW theorem. This latter is a polynomial ring (over (a basis of) $L^*$), and hence an integral domain. Now choose nonzero $a, b \in U(L)$ as above. Then $gr_F(ab) = gr_F(a) gr_F(b) \neq 0$ in $gr_F^{n+m}(U(L)) \subseteq \mathrm{Sym}(L)$, so $ab \neq 0$ in $U(L)$ by the PBW isomorphism. $\square$

Next, we systematically show the Noetherian-ness of $U(L)$. In the rest of this section, all ideals and modules over a (possibly non-commutative) ring are left-ideals and left-modules, respectively.

We begin by recalling (with details) some basics on Noetherian rings and modules.

**Definition 2.1.** Let $R$ be a (possibly non-commutative) ring and $M$ a left $R$-module (e.g. $R = U(L)$). We say $M$ is *Noetherian* if it satisfies the *ascending chain condition (a.c.c.)*: every non-decreasing infinite chain of $R$-submodules of $M$,

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

eventually stabilizes / is constant.

A ring $R$ is *Noetherian* if it is Noetherian as an $R$-module.

A basic lemma on Noetherian-ness is as follows.

**Lemma 2.2.** *The following are equivalent for an $R$-module $M$:*

(1) *$M$ satisfies the a.c.c., i.e. is Noetherian.*
(2) *Every nonempty set $S$ of submodules of $M$ has a maximal element.*
(3) *Every submodule of $M$ is finitely generated.*

This shows for instance that every PID $R$ is Noetherian, since every submodule of $R$ (i.e., ideal) is singly generated.

*Proof.* (1) $\implies$ (2): Let $M_1 \leq M$ be any submodule in $S$. If it is not maximal in $S$, there exists $M_2 \in S$ that strictly contains $M_1$. Proceeding inductively, we obtain a strictly increasing chain of submodules $M_1 \subsetneq M_2 \subsetneq \cdots$ of $M$ – which contradicts (1). So, this chain must instead stabilize, and so the final element in it is maximal in $S$.

(2) $\implies$ (3): If the submodule $N$ of $M$ equals 0 then it is finitely generated. Else let $S$ denote the set of finitely generated nonzero submodules of $N$; this is nonempty and hence has a maximal element $N_{\max}$, say. If $N_{\max} \subsetneq N$, then let $m \in N \setminus N_{\max}$; then $N_{\max} + Rm$ is a strictly larger, finitely generated submodule of $N$, which is impossible. Hence $N = N_{\max} \in S$ is finitely generated.

(3) $\implies$ (1): Let $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be an increasing chain of submodules of $M$. Set $M' := \cup_{j \geq 0} M_j$; by (3) this is finitely generated, say by generators $m_1, \dots, m_k$. Now each $m_j \in M_{n_j}$ for some $n_j \geq 0$, so letting $n := \max_j n_j \geq 0$, we obtain $m_j \in M_n \ \forall j$. But then

$$M' = \sum_{j=1}^{k} Rm_j \subseteq M_n \subseteq M_{n+1} \subseteq \cdots \subseteq M',$$

so the chain stabilizes at $M_n = M'$. $\square$

Now our strategy to solve Question 4, is to proceed along the following steps:

(1) If $A$ is a left-Noetherian ring, then so is the polynomial ring $A[x]$. Thus, every polynomial algebra in finitely many variables over any field is Noetherian. (This is / follows from the famous *Hilbert basis theorem*.)
(2) If $R$ is a filtered ring / $\mathbb{F}$-algebra such that $gr(R)$ is Noetherian, then so is $R$.
(3) If $R$ is Noetherian then every finitely generated module over it is Noetherian.
(4) (This is already done in Lemma 2.2.) An $R$-module $M$ is Noetherian if and only if every submodule of $M$ is finitely generated.

This shows Question 4. Moreover, for a complex semisimple Lie algebra $\mathfrak{g}$, setting $R = U(\mathfrak{g})$ and choosing any object $M$ of Category $\mathcal{O}$, it follows that $M$ is finitely generated and hence Noetherian. Thus, every submodule of $M$ is finitely generated, hence in $\mathcal{O}$ as well (since we had seen that it satisfies the other axioms/conditions (O2) and (O3)).

**Step 1:** We now implement the above steps, starting with:

*Proof of Hilbert basis theorem (taken from Wikipedia).* We proceed by contradiction. Suppose – by Lemma 2.2 – that $\mathfrak{a} \subseteq A[x]$ is a left-ideal that is not finitely generated. Choose $0 \neq f_0 \in \mathfrak{a}$ of least degree. Inductively, given $f_0, \dots, f_{n-1}$ for $n \geq 1$, define the ideal $\mathfrak{b}_n := \sum_{j=0}^{n-1} A[x] f_j$. Then $\mathfrak{b}_n \subsetneq \mathfrak{a}$; now choose $f_n \in \mathfrak{a} \setminus \mathfrak{b}_n$ of smallest degree. Then $0 \leq \deg(f_0) \leq \deg(f_1) \leq \cdots$.

Let $a_n$ be the leading coefficient of $f_n$ for each $n \geq 0$, and let $\mathfrak{c}_n \subseteq A$ be the left-ideal in $A$ generated by $a_0, a_1, \dots, a_{n-1}$. As $A$ is Noetherian, the increasing chain of ideals $(\mathfrak{c}_n)_n$ stabilizes, i.e. $\mathfrak{c}_N = \mathfrak{c}_{N+1} = \cdots$ for some $N \geq 0$. Now write

$$a_N = \sum_{j=0}^{N-1} u_j a_j, \qquad u_j \in A$$

and define

$$g(x) := \sum_{j=0}^{N-1} u_j x^{\deg(f_N) - \deg(f_j)} f_j \in \mathfrak{b}_N.$$

Then $\deg(g) \leq \deg(f_N) =: d$, say; and the coefficient of $x^d$ is $\sum_{j=0}^{N-1} u_j a_j = a_N$, which is the leading coefficient of $f_N \in \mathfrak{a} \setminus \mathfrak{b}_N$. From this analysis it follows that $f_N - g$ is a polynomial in $\mathfrak{a} \setminus \mathfrak{b}_N$, of degree $< \deg(f_N)$. This contradicts the choice of $f_N$, and concludes the proof. $\square$

Thus, $gr(R) \cong \mathrm{Sym}(L)$ is Noetherian if $L$ is finite-dimensional.

**Step 2:** We next explain why a filtered ring $R$ is Noetherian if $gr(R)$ is (e.g. $R = U(L)$). We work more generally: a unital ring $R$ is *filtered* if there is an increasing chain of abelian subgroups

$$0 = F_{-1}(R) \subsetneq F_0(R) \subseteq \cdots, \qquad 1 \in F_0(R), \ \bigcup_{n \geq 0} F_n(R) = R;$$

and for such a ring, a filtered $R$-module is an $R$-module $M$ equipped with a filtration

$$0 = F_{-1}(M) \subseteq F_0(M) \subseteq \cdots, \qquad \bigcup_{m \geq 0} F_m(M) = M$$

which is "compatible" with $R$ in that $F_n(R) \cdot F_m(M) \subseteq F_{n+m}(M)$ for all $n, m \geq 0$. Corresponding to these filtrations on $R, M$, we construct their associated graded algbras:

$$gr_F(R) := \bigoplus_{n \geq 0} F_n(R)/F_{n-1}(R), \qquad gr_F(M) := \bigoplus_{m \geq 0} F_m(M)/F_{m-1}(M).$$

Note for all $m \in M$ that $gr_F(m)$ is homogeneous (i.e., lives in a single graded component of $gr_F(M)$). Moreover, $gr_F(M)$ is a $gr_F(R)$-module via: $gr_F^n(R) \cdot gr_F^m(M) \subseteq gr_F^{n+m}(M)$.

We can now show the desired result:

**Proposition 2.3.** *Let $R$ be a filtered ring, and $M$ a filtered $R$-module as above.*

1. *Suppose $M, N$ are filtered modules as above, and $\phi : M \to N$ is a filtered $R$-module map sending $F_m(M) \to F_m(N)$ for all $m \geq 0$. Then $\phi$ induces a map of graded modules $gr_F(\phi) : gr_F(M) \to gr_F(N)$. Moreover, if $gr_F(\phi)$ is a surjection, then so is $\phi$.*
2. *If $gr_F(M)$ is a finitely generated $gr_F(R)$-module, then $M$ is a finitely generated $R$-module.*
3. *If $gr_F(R)$ is Noetherian, then so is $R$.*

*Proof.*

(1) Define $gr_F(\phi)$ as follows: given $0 \neq \overline{m} \in gr_F^p(M)$, choose any lift $m \in F_p(M) \setminus F_{p-1}(M)$ and define

$$gr_F(\phi)(\overline{m}) = gr_F(\phi)(m + F_{p-1}(M)) := \phi(m) + F_{p-1}(N).$$

This is independent of the choice of lift $m$, since if $m'$ is another lift then $m - m' \in F_{p-1}(M)$, so

$$gr_F(\phi)(\overline{m}) - gr_F(\phi)(\overline{m'}) = \phi(m) - \phi(m') + F_{p-1}(N) = F_{p-1}(N) = \overline{0}.$$

Moreover, $gr_F(\phi)$ is a $gr_F(R)$-module map, because given $r \in F_n(R) \setminus F_{n-1}(R)$ and $m \in F_p(M) \setminus F_{p-1}(M)$ with $n, p \geq 0$, we have $\phi(m) \in F_p(N)$ and so

$$r \cdot m \in F_{n+p}(M), \qquad r \cdot \phi(m) \in F_{n+p}(N).$$

But then

$$\begin{aligned} gr_F(\phi)(gr_F(r) \cdot gr_F(m)) &= \phi(rm) + F_{n+p-1}(N), \\ gr_F(r) \cdot gr_F(\phi)(gr_F(m)) &= r\phi(m) + F_{n+p-1}(N), \end{aligned}$$

and these are equal elements of $gr_F^{n+p}(N)$, by the definitions.

It remains to show that if $gr_F(\phi)$ is onto then so is $\phi$. Given $n_0 \in F_p(N) \setminus F_{p-1}(N)$, choose $m_0 \in F_p(M) \setminus F_{p-1}(M)$ such that $gr_F(\phi)(m_0 + F_{p-1}(M)) = n_0 + F_{p-1}(N)$. If $n_0 - \phi(m_0) = 0$ then we are done; else $n_0 - \phi(m_0) \in F_{p'}(N) \setminus F_{p'-1}(N)$ for some $p' < p$. Repeat this procedure for $n_1 := n_0 - \phi(m_0)$. This keeps reducing the value of $p$, and after finitely many steps we are done.

(2) Suppose $gr_F(M)$ is generated by generators $0 \neq \overline{m}_1, \dots, \overline{m}_k$, which we can take to be homogeneous, say of degree $n_j$ for each $j$. Now fix any lifts $m_j \in F_{n_j}(M) \setminus F_{n_j-1}(M)$ of $\overline{m}_j$, for all $j$.

For any integer $n \in \mathbb{Z}$ and $R$-module $M$, define a filtration $F^{(n)}$ on $M$ via: $F_m^{(n)}(M) := F_{n+m}(M)$ if $n \geq -m$, and 0 otherwise. Now define the map $\phi_j : F^{(-n_j)}(R) \to M$, sending $r \mapsto r \cdot m_j$. This is a morphism of filtered $R$-modules that sends $F_n(R) = F_{n+n_j}^{(-n_j)}(R) \to F_{n+n_j}(M)$ for all $n \geq 0$. Thus, we obtain a morphism of filtered $R$-modules

$$\sum_{j=1}^k \phi_j : \bigoplus_{j=1}^k F^{(-n_j)}(R) \to M,$$

which in turn induces by (1) a map of graded modules, whose image contains $gr_F(m_j) = \overline{m}_j$. Hence the graded map is onto, and hence so is $\sum_j \phi_j$ by (1). But then $M$ is finitely generated.

(3) This is a special case of part (2). Each ideal $M = I$ of $R$ is filtered via: $F_m(I) = I \cap F_m(R)$. So $gr_F(I)$ is an ideal of $gr_F(R)$, hence finitely generated; thus so is $I$. □

**Step 3:** Finally, we show:

**Proposition 2.4.** *Let $R$ be a Noetherian ring.*

(1) *Given a short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \xrightarrow{\pi} 0$ of $R$-modules, $M$ is Noetherian if and only if $M', M''$ are Noetherian.*

(2) *Every finitely generated $R$-module is Noetherian.*

*Proof.*

(1) First suppose $M$ is Noetherian. By Lemma 2.2, every submodule of $M'$, being a submodule of $M$, is finitely generated and so $M'$ is Noetherian. Similarly, let $N'' \leq M''$ be a submodule. Then it lifts to a submodule $\widetilde{N''} \leq M$, which is finitely generated. The image of this (finite) generating set now generates $N''$, so $M''$ is Noetherian.

Conversely, suppose $M', M''$ are Noetherian, and let $N \leq M$ be a submodule. Let $N' := N \cap M' \leq M'$ and $N'' := \pi(N) \leq M''$; then both of these are finitely generated by hypothesis – say by $m'_1, \dots, m'_k$ and $m''_1, \dots, m''_l$ respectively. Lift $m''_j$ to $\widetilde{m''_j} \in N$; then we claim that the $m'_i$ and $\widetilde{m''_j}$ together generate $N$ – which would show $N$ is finitely generated and hence that $M$ is Noetherian (by Lemma 2.2). Indeed, given $m \in N$, $\pi(m) \in N''$ and so is of the form $\sum_{j=1}^l r_j m''_j$. Hence

$$m - \sum_{j=1}^{l} r_j \widetilde{m''_j} \in \ker(\pi) \cap N = N',$$

and so it is another $R$-linear combination $\sum_{i=1}^k r'_i m'_i$. Thus,

$$m = \sum_{i=1}^{k} r'_i m'_i + \sum_{j=1}^{l} r_j \widetilde{m''_j},$$

as desired.

(2) Suppose $M$ is finitely generated, say $M = Rm_1 + \cdots + Rm_k$. Then the map : $R^{\oplus k} = R\mathbf{e}_1 \oplus \cdots \oplus R\mathbf{e}_k \to M$, sending $\mathbf{e}_j$ to $m_j$, is a surjective $R$-module map. Inductively "building up" the free module $R^{\oplus k}$ via split short exact sequences, the preceding part yields that $R^{\oplus k}$ is Noetherian; then it yields that its quotient $M$ is Noetherian. □

## 3. Days 3–5: 18–20 Dec, 2024 – see Shushma Rani's scribed notes

## 4. Day 6: 21 Dec, 2024 – Ext's in $\mathcal{O}$ are finite-dimensional; projectives in $\mathcal{O}$

**4.1. Ext-spaces in $\mathcal{O}$ are finite-dimensional.** We now complete the proof of why all vector spaces $\mathrm{Ext}^1_{\mathcal{O}}(M, M')$ are finite-dimensional. As mentioned in the lecture, since $\mathcal{O}$ is finite length, by considering the long exact sequences of $\mathrm{Ext}^1_{\mathcal{O}}$'s we reduce to the case of $M = L(\lambda), M' = L(\mu)$ simple modules. (Even here, we had shown that unless $\chi_\lambda = \chi_\mu$, i.e. $\mu \in W \bullet \lambda$, this $\mathrm{Ext}^1$-space is zero.) We now work with general $\lambda, \mu$, and show:

**Theorem 4.1.** *For $\lambda \in \mathfrak{h}^*$, define $N_{\max}(\lambda)$ via: $0 \to N_{\max}(\lambda) \to M(\lambda) \to L(\lambda) \to 0$. Now given arbitrary $\lambda, \mu \in \mathfrak{h}^*$, we have $\mathrm{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) \neq 0$ if and only if either*

1. *$\mu > \lambda$ and $N_{\max}(\mu) \twoheadrightarrow L(\lambda)$ (so, $L(\mu), L(\lambda)$ are the top two terms in a Jordan–Hölder series for $M(\mu)$);*
2. *or $\lambda > \mu$ and the (non-split) extension is the restricted dual $F(\cdot)$ applied to the preceding short exact sequence with $\mu, \lambda$ switched.*

*In either case, $\mathrm{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda))$ is finite-dimensional.*

*Proof.* **Step 1:** Recall that we showed in the lectures: there are no non-split extensions between highest weight modules whose highest weights have distinct central characters. Here we first show that one cannot extend a simple module $L(\lambda)$ by a highest weight module $M(\lambda) \twoheadrightarrow V$, i.e.:

$$M(\lambda) \twoheadrightarrow V \qquad \Longrightarrow \qquad \operatorname{Ext}^1_{\mathcal{O}}(V, L(\lambda)) = 0.$$

To see this, suppose $0 \to L(\lambda) \to M \xrightarrow{\pi} V \to 0$ in $\mathcal{O}$. By character theory, $M_\lambda$ is the highest weight space, of dimension 2. Choose $m_\lambda \in M_\lambda \setminus L(\lambda)_\lambda$ and $0 \neq v_\lambda \in V_\lambda$. By rescaling $v_\lambda$, we may assume that $\pi(m_\lambda) = v_\lambda$. Now $U(\mathfrak{g})m_\lambda \cap L(\lambda) = U(\mathfrak{n}^-)m_\lambda \cap L(\lambda)$ is a submodule of $L(\lambda)$ which does not contain $L(\lambda)_\lambda$. As $L(\lambda)$ is simple, $U(\mathfrak{g})m_\lambda \cap L(\lambda) = 0$. Hence the sum $L(\lambda) + U(\mathfrak{g})m_\lambda$ in $M$ is direct, so that

$$\operatorname{char} L(\lambda) + \operatorname{char} U(\mathfrak{g})m_\lambda \leq \operatorname{char} M = \operatorname{char} L(\lambda) + \operatorname{char} V.$$

Thus $\operatorname{char} U(\mathfrak{g})m_\lambda \leq \operatorname{char} V$. But $\pi(m_\lambda) = v_\lambda$ and both are maximal vectors, so $U(\mathfrak{g})m_\lambda \twoheadrightarrow V$ and hence $\operatorname{char} U(\mathfrak{g})m_\lambda \geq \operatorname{char} V$. Thus the modules have equal characters, and the surjection implies they are isomorphic, as claimed.

**Step 2:** Now we are ready to show the result. We first show that if the conditions in (1) hold then $\operatorname{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) \neq 0$. Indeed, if we define $N$ via $0 \to N \to N_{\max}(\mu) \to L(\lambda) \to 0$, then the $\operatorname{Ext}^1$-group vanishing implies $M(\mu)/N \cong L(\mu) \oplus L(\lambda)$. But as $\mu \neq \lambda$ from above, this implies that $M(\mu)$ has two non-isomorphic simple quotients, which is false.

Next, if the conditions in (2) hold, and the $\operatorname{Ext}^1$-group vanishes, then every short exact sequence of the given form splits, in which case its restricted dual sequence also splits – and this contradicts the preceding paragraph. This shows one implication.

**Step 3:** Conversely, given a non-split short exact sequence $0 \to L(\lambda) \to M \to L(\mu) \to 0$, we showed above that $\mu \neq \lambda$. Let $m_\mu \in M_\mu$ be a lift of a fixed nonzero maximal vector $v_\mu \in L(\mu)_\mu$. Then $\mathfrak{n}^+ \cdot m_\mu$ is either zero or not.

(1) If $\mathfrak{n}^+ m_\mu = 0$, then $U(\mathfrak{n}^-)m_\mu = U(\mathfrak{g})m_\mu$ is a highest weight module, hence surjects onto its unique simple quotient $L(\mu)$. Moreover, $U(\mathfrak{g})m_\mu \cap L(\lambda)$ is a submodule of $L(\lambda)$ and hence must be $L(\lambda)$ since the extension is non-split by assumption. In particular, $\lambda < \mu$, and $L(\lambda) \subsetneq U(\mathfrak{g})v_\mu \twoheadrightarrow L(\mu)$. From this it follows that $N_{\max}(\mu) \twoheadrightarrow L(\lambda)$.

    Moreover, the module structure / extension is completely determined by the image in $M$ of the highest weight vector of $L(\lambda)$. But this necessarily lies in $M_\lambda$, which is a vector space of dimension at most $KPF(\mu - \lambda)$, where $KPF(\cdot)$ denotes the Kostant partition function. Thus $\dim \operatorname{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda)) < \infty$ in this case.

(2) The other case is that $\mathfrak{n}^+ m_\mu \neq 0$. Then it still maps into $L(\lambda)$, in which case $\mu < \lambda$. Now this short exact sequence does not split, hence its dual sequence does not either (else we could again apply $F$ and get a contradiction):

$$0 \to L(\mu) \to F(M) \to L(\lambda) \to 0.$$

    But $\mu < \lambda$, so the above analysis implies that the preceding case must hold. $\square$

## 4.2. **Basics on projective modules.**

**Definition 4.2.** An object $P$ in an abelian category $\mathcal{A}$ is *projective* if given an epimorphism $f : N \twoheadrightarrow M$ and a morphism $g : P \to M$ in $\mathcal{A}$, there exists a (not necessarily unique) morphism $h : P \to N$ such that $h \circ f = g$.

There are several equivalent notions to this; we use two of them in today's lecture.

**Proposition 4.3.** *Suppose $\mathcal{A}$ is an abelian category of $R$-modules (over some associative ring $R$). Then the following are equivalent for an object $P$:*

1. *$P$ is projective in $\mathcal{A}$.*
2. *$\mathrm{Ext}^1_{\mathcal{A}}(P, N) = 0$ for all objects $N$.*
3. *$\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor from $\mathcal{A}$ to abelian groups.*

In what follows, we suppress the subscript $\mathcal{A}$ from all Hom and Ext spaces.

*Proof.* We show a cyclic chain of implications.

(1) $\implies$ (2): Suppose $0 \to N_0 \to N \xrightarrow{\pi} P \to 0$ in $\mathcal{A}$. Apply (1) with $M$ replaced by $P$ and $g : P \to P$ the identity map; thus, there exists $h : P \to N$ with $\pi \circ h = \mathrm{id}_P$. We now claim in $N$ that $\mathrm{im}(h) \cap \ker(\pi) = 0$, since if $h(p) \in \ker(\pi) = N_0$ then $p = (\pi \circ h)(p) = 0$. Moreover, $\ker(h) = 0$ for the same reason. But then one can show the short exact sequence splits, since one has

$$P \oplus N_0 \cong \mathrm{im}(h) \oplus \ker(\pi) \subseteq N,$$

(2) $\implies$ (3): Apply $\mathrm{Hom}_{\mathcal{A}}(P, -)$ to a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$, to get the long exact sequence

$$0 \to \mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M'') \to \mathrm{Ext}^1(P, M') \to \cdots$$

The final term here is zero, which yields (3).

(3) $\implies$ (1): Say $N, M, f, g$ are as in the definition. Let $L = \ker(f) \subseteq N$, so that $0 \to L \to N \xrightarrow{f} M \to 0$. Applying $\mathrm{Hom}(P, -)$ yields the short exact sequence (by (3)):

$$0 \longrightarrow \mathrm{Hom}(P, L) \longrightarrow \mathrm{Hom}(P, N) \xrightarrow{\pi} \mathrm{Hom}(P, M) \to 0,$$

where the map $\pi$ is simply post-composition with $f$. As $\pi$ is a surjection, there exists $h \in \mathrm{Hom}(P, N)$ in the preimage of $g \in \mathrm{Hom}(P, M)$, which yields (1). □

An application of these equivalent conditions is:

**Corollary 4.4.** *Two objects $P', P''$ in $\mathcal{A}$ are projective if and only if $P' \oplus P''$ is.*

*Proof.* Pick any object $M$ in $\mathcal{A}$ and apply $\mathrm{Hom}(-, M)$ to the split short exact sequence $0 \to P' \to P' \oplus P'' \to P'' \to 0$, to get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(P'', M) \to \mathrm{Hom}(P' \oplus P'', M) \to \mathrm{Hom}(P', M) \to& \\ \to \mathrm{Ext}^1(P'', M) \to \mathrm{Ext}^1(P' \oplus P'', M) \to \mathrm{Ext}^1(P', M) \to \cdots& \end{aligned}$$

Notice that as the short exact sequence is split, the first three terms are already exact.

We now prove both implications. The "only if" part follows from the second row of the long exact sequence, since $\mathrm{Ext}^1(P', M) = \mathrm{Ext}^1(P'', M) = 0$ by the proposition. For the "if" part, by the proposition we have $\mathrm{Ext}^1(P' \oplus P'', M) = 0$, which combined with the exactness of the first row yields: $\mathrm{Ext}^1(P'', M) = 0$. Reversing the roles of $P'$ and $P''$, we are done. □