MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Wednesday, August 14* in my office hours, or previously in class)

Question 1. The goal here is to show that the set of complex numbers
$$\mathbb{C} := \{a + bi = a + b\sqrt{-1} : a, b \in \mathbb{R}\}$$
under the operations
$$(a + bi) + (c + di) := (a + c) + (b + d)i, \quad (a + bi) \cdot (c + di) := (ac - bd) + (ad + bc)i,$$
$$0 := 0 + 0i, \quad 1 := 1 + 0i,$$
$$-(a + bi) := (-a) + (-b)i, \quad (a + bi)^{-1} := \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$
is a field. (You are allowed to use that $\mathbb{R}$ is a field.)

I felt you should do some of these verifications at least once in your life – and it is not likely you will get to do this in any other course – so here, **prove that**:

    (1) Multiplication is associative.
    (2) $z \cdot z^{-1} = 1$ for all nonzero complex z.
    (3) The distributive law holds.

Question 2. Suppose $\mathbb{F}$ is a field, with $a, b \in \mathbb{F}$. Prove the following statements.

    (1) The elements a^{-1} (for $a \neq 0$) and $-a$ are unique in $\mathbb{F}$.
    (2) $-a = (-1) \cdot a$.
    (3) $(-1)^2 = 1$.
    (4) $ab = 0$ in $\mathbb{F}$, if and only if $a = 0$ or $b = 0$.

Question 3. Prove that a nonempty subset W of an $\mathbb{F}$-vector space V is a subspace if and only if $cw + w' \in W$ for all $w, w' \in W$ and $c \in \mathbb{F}$.

Question 4. Suppose V is a vector space over a field $\mathbb{F}$, and $c \in \mathbb{F}, v \in V$. Prove that $c \cdot \mathbf{0} = \mathbf{0} = 0 \cdot v$, where 0 is the zero in $\mathbb{F}$ and $\mathbf{0}$ is the zero in V.

Question 5. In class, we saw that the set of functions from any set to a field is a vector space. Now let us see why **every** vector space is a subspace of such a space of functions.

Let B be a nonempty set, and $\mathbb{F}$ a field. Define $Fun_0(B, \mathbb{F})$ to be the set of all functions $f : B \to \mathbb{F}$ such that $f(b) = 0$ for all but finitely many $b \in B$. Show that $Fun_0(B, \mathbb{F})$ is an $\mathbb{F}$-vector space.

Note: You are allowed to use the fact mentioned in class, that the set of *all* functions $Fun(B, \mathbb{F})$ is an $\mathbb{F}$-vector space, which might help bypass a lot of the routine verifications for the subset Fun_0.

(Later, we will say that every $\mathbb{F}$-vector space V has a "basis" B, and then the space V can in fact be identified precisely with our space here: $Fun_0(B, \mathbb{F})$.)