MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Thursday, August 22* in TA's office hours, or previously in class)

Question 1. Given any field $\mathbb{F}$, prove the following formula for its prime subfield:

$$\mathbb{F}_{\text{prime}} = \bigcap_{\mathbb{K} \subseteq \mathbb{F} \text{ subfield}} \mathbb{K}.$$

Question 2. Prove that:

(1) $(AB)^T = B^T A^T$ for all $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times p}$ – by checking entry by entry.

(2) Whenever both sides are defined, $(B+C)A = BA+CA$ for $B, C \in \mathbb{F}^{m \times n}, A \in \mathbb{F}^{n \times p}$ – *without* using any entry-by-entry calculations, but using the previous part and that $A(B+C) = AB + AC$ (shown – or at least, stated – in class).

Question 3. The *trace* of a square matrix $A = (a_{ij})_{i,j=1}^n$ is the sum of its diagonal entries: $a_{11} + a_{22} + \cdots + a_{nn}$. Given integers $m, n \geq 1$ and matrices $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times m}$, prove that AB and BA have the same trace, even if they have different sizes.

Question 4. Suppose $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times p}$ are invertible. Prove that A^T and AB are invertible, by showing that $(A^T)^{-1} = (A^{-1})^T$ and $(AB)^{-1} = B^{-1}A^{-1}$. (Prove only from one side, e.g. $M \cdot M^{-1} = \text{Id}$, not from both sides.)

Question 5. For each of the following, explain whether or not the specified subset (of the corresponding vector space) is a subspace.

(1) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(1) - f(2) + 4f(3) = 0$.

(2) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(3) = f(2) + 1$.

(3) The subset of solutions to $Ax = b$ for some vector $b \neq 0$. Here $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ for some integers $m, n \geq 1$.