

## MA219 – Linear Algebra 2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

**Homework Set 3** (*due by Thursday, September 5* in TA's office hours, or previously in class, or in the Instructor's mailbox by 5pm)

Throughout this homework (and this course),  $\mathbb{F}$  denotes an arbitrary field.

**Question 1.** Solve the following systems of linear equations. Use row operations to obtain the RREF in all cases.

- (1)  $x + 2y + 3z = 0$ ,  $x + y + z = 1$ ,  $-x + z = 1$ , over a field of characteristic 0.
- (2)  $x + 4y + 5z = 1$ ,  $y - z = 3$ ,  $x + z = 5$ , over a field of characteristic 7.

**Question 2.** Suppose  $\mathbb{F}$  is a field, and  $n \geq 1$  an integer. For integers  $1 \leq i, j \leq n$ , define the  $n \times n$  matrix  $E_{ij}$  as having all entries zero, except 1 in the  $(i, j)$ -entry.

Now find the  $\mathbb{F}$ -span of the following sets – give (with some justification – maybe via the explicit description) the “conceptual description” (see e.g. towards the end of Lecture L06 in the videos).

- (1) The matrices  $E_{ii}$  for  $1 \leq i \leq n$ .
- (2) The matrices  $E_{ij}$  for  $i < j$ .
- (3) The polynomials  $x^2 - x, x^3 - x^2, \dots$  and the polynomial  $x$ , with  $\mathbb{F} = \mathbb{R}$ .

**Question 3.** Suppose  $V$  is a vector space over  $\mathbb{F}$ , and  $L \subset V$  is a linearly independent subset. If  $v \in V$  is not in the span of  $L$ , show that  $L \cup \{v\}$  is also linearly independent.

**Question 4.** Let  $\mathbb{F} = \mathbb{Q}$ .

- (1) Show that  $\mathbb{R}$  is not a finite-dimensional  $\mathbb{Q}$ -vector space.
- (2) Suppose  $V$  is a *countable*-dimensional  $\mathbb{Q}$ -vector space, i.e. a vector space with a countably infinite basis  $v_1, v_2, \dots, v_n, \dots$ . Show that  $V$  is the union of its finite-dimensional subspaces  $V_n$  spanned by  $v_1, \dots, v_n$ .
- (3) Show that  $\mathbb{R}$  is not a countable-dimensional  $\mathbb{Q}$ -vector space.

**Question 5.**

- (1) Show that three vectors in  $\mathbb{R}^2$  are linearly dependent.
- (2) Find three vectors in  $\mathbb{R}^2$  such that any two are linearly independent.

**Question 6.**

- (1) Suppose  $S_1, S_2$  are subsets of an  $\mathbb{F}$ -vector space. Prove that  $S_1, S_2$  have the same spans if and only if each set is contained in the span of the other.
- (2) The *row space* of a matrix is the span of its rows. If  $A, B \in \mathbb{F}^{m \times n}$  are row-equivalent, prove that their row spaces are equal.