MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 5 (*due by Thursday, September 26* in TA's office hours, or previously in class, or in the Instructor's mailbox by 5pm)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. We have seen that (by convention,) the zero vector space over any field has exactly one basis: the empty set. Now:

(1) Classify all nonzero vector spaces over all fields, which also have exactly one unordered basis – i.e., exactly one basis up to permuting its basis elements.

(2) Classify all nonzero vector spaces over all fields, which have exactly two unordered bases.

Question 2. Suppose $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $\theta \in \mathbb{R}$. Suppose $T : V \to V$ is the linear transformation that rotates a vector counterclockwise by θ (radians). Compute the matrix of T with respect to the standard basis of V.

Question 3. Suppose $\mathbb{F}$ is a field, and $T : \mathbb{F}^2 \to \mathbb{F}^2$ is the linear operator $T(x_1, x_2) := (x_2, -x_1)$, where $(x_1, x_2)^T = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ is with respect to the standard ordered basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$.

(1) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}}$?

(2) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}'}$, where $\mathcal{B}' = (\mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1)$?

(3) What is the transition matrix of $\mathcal{B}'$ into $\mathcal{B}$? Meaning, find the matrix P such that $[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$ for all $v \in \mathbb{F}^2$.

(4) Suppose $\mathbb{F}$ has characteristic not 2 (so $2 = 1+1$ in $\mathbb{F}$). What is the coordinate vector of $(-1, 2)^T$ in the standard basis, when written out in the basis $\mathcal{B}'$?

Question 4. Suppose $\mathbb{F}$-vector spaces V, W, X have ordered bases $(v_1, \dots, v_n)$, $(w_1, \dots, w_m)$, and $(x_1, \dots, x_p)$ for positive integers n, m, p respectively. Write down a basis of the vector space of bilinear maps : $V \times W \to X$, and prove that it is a basis.

Question 5. Recall that the *direct product* of a family $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces is their Cartesian product, denoted

$$\prod_{i\in I} V_i = \times_{i\in I} V_i,$$

with a typical element $(v_i)_{i\in I}$. Also fix the *projection* maps

$$\pi_{i_0} : \prod_{i\in I} V_i \to V_{i_0}, \qquad (v_i)_{i\in I} \mapsto v_{i_0}.$$

(1) Verify that each π_{i_0} is a surjective $\mathbb{F}$-linear map.
(2) Write out the (complete) proof that this product satisfies the following "universal property":

Given any $\mathbb{F}$-vector space Z, and $\mathbb{F}$-linear maps $\varphi_i : Z \to V_i$ for all $i \in I$, there exists a unique $\mathbb{F}$-linear map $\varphi : Z \to \prod_{i\in I} V_i$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$.

In other words, the Cartesian product proves the existence of an object that satisfies this universal property. (By class, every other "candidate" is isomorphic to this one.)