MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 6 (*due by Thursday, October 10* in TA's office hours, or or in the Instructor's mailbox by 5pm, or previously)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Prove that $\mathbb{R}^2/\mathbb{R}\mathbf{e}_1 \cong \mathbb{R}\mathbf{e}_2$ as $\mathbb{R}$-vector spaces.

Question 2. Here we see the difference between linear maps and bilinear maps – which is at the heart of why the coproduct (= direct sum / Cartesian product) and tensor product of two vector spaces have different dimensions.

Suppose V, W are two finite-dimensional vector spaces over $\mathbb{F}$, with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ respectively.

(1) Prove that $\mathrm{Lin}(V \times W, \mathbb{F})$ has a basis given by the $n + m$ linear maps, which take one of the elements $\{v_i \oplus 0, 0 \oplus w_j\}$ to 1 and the others to zero (and extending by linearity).

(2) Prove that the $n \cdot m$ maps indexed by $i_0 \in [1, n]$ and $j_0 \in [1, m]$, given by

$$\varphi_{i_0,j_0}\left(\sum_{i=1}^{n} c_i v_i, \sum_{j=1}^{m} d_j v_j\right) := c_{i_0} d_{j_0}$$

form an $\mathbb{F}$-basis of $\mathrm{BiLin}(V \times W, \mathbb{F})$.

Question 3. Suppose an $\mathbb{F}$-vector space V has an ordered basis $(v_1, \ldots, v_n)$. For $1 \leq i_0 \leq n$, define the *dual functionals* $\varphi_{i_0} : V \to \mathbb{F}$ via:

$$\varphi_{i_0}\left(\sum_{i=1}^{n} c_i v_i\right) := c_{i_0}.$$

Assuming that $\varphi_{i_0} \in V^*$, show that these vectors form a basis of V^*. (This is called the *dual basis*.)

Question 4. Recall the direct product and direct sum (or coproduct) of a set $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces, constructed in class (and studied in the preceding homework set). The goal of this exercise is to show that

$$\left(\bigoplus_{i\in I} V_i\right)^* \cong \prod_{i\in I} V_i^*,$$

by going the 'reverse' way:

(1) Given $\mathbf{\Phi} = (\varphi_i)_{i\in I}$, with each $\varphi_i \in V_i^*$, first show that $(\varphi_i)_{i\in I}$ yields a linear map from $\bigoplus_{i\in I} V_i$ to $\mathbb{F}$. Let us call this map $T(\mathbf{\Phi})$.
(2) Show that the assignment $T : \mathbf{\Phi} \mapsto T(\mathbf{\Phi})$ is a linear map, from $\prod_{i\in I} V_i^*$ to $\left(\bigoplus_{i\in I} V_i\right)^*$.
(3) Show that T is one-to-one and onto.

Question 5. (*Quotient spaces – you don't need to submit this solution.*) Suppose V is an $\mathbb{F}$-vector space, and $W \subset V$ a subspace. Define a relation $v \sim v'$ on V if $v - v' \in W$. Now define the quotient space V/W to be the set of distinct equivalence classes $[v]$, under the relation:

$$a[v] + b[v'] := [av + bv'], \qquad a, b \in \mathbb{F},\ v, v' \in V.$$

(1) Verify that $\sim$ above is an equivalence relation on V.
(2) Check that the addition defined above (setting $a = b = 1$) is associative.
(3) Check that $\mathbf{0}_{V/W} := [w]$ for $w \in W$ is an (or 'the') additive identity in V/W.
(4) Assuming that V/W is a vector space under the above operations (which it is!), check that the map $\pi : V \to V/W$ sending $v \in V$ to $[v]$ is a surjective $\mathbb{F}$-linear map.
(5) Prove that the quotient space satisfies the following 'universal property':

Given any $\mathbb{F}$-vector space Z, and a $\mathbb{F}$-linear map $\varphi : V \to Z$ which maps W to $\mathbf{0}_Z$, there exists a unique $\mathbb{F}$-linear map $\overline{\varphi} : V/W \to Z$ such that $\varphi = \overline{\varphi} \circ \pi$ (with π as in the previous part).