MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 7 (*due by Thursday, October 24* in TA's office hours, or previously in class, or in the Instructor's mailbox by 5pm)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Using results from class about how the determinant changes under elementary row operations (or other results about the determinant), compute the determinants of the matrices $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Question 2. Given a square matrix $A \in \mathbb{F}^{n\times n}$, define its *adjugate* matrix $adj(A) \in \mathbb{F}^{n\times n}$ to have (i,j) entry $(-1)^{i+j} \det A_{j|i}$, where $A_{j|i} \in \mathbb{F}^{(n-1)\times(n-1)}$ is the matrix obtained by removing the jth row and ith column of A. Prove the following properties for any matrix $A \in \mathbb{F}^{n\times n}$, say with $n \geq 2$:

(1) $adj(A) \cdot A = A \cdot adj(A) = (\det A)\mathrm{Id}_n$.
(2) If A is singular then $adj(A)$ is also singular.
(3) $\det(adj A) = (\det A)^{n-1}$.
(4) $adj(A^T) = adj(A)^T$.

Question 3. A *Vandermonde* matrix is a matrix of the form

$$M_{n\times n} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

where $n \geq 1$ is an integer, and $a_1, \dots, a_n \in \mathbb{F}$ are scalars.

Prove (e.g. by induction on n) that if $n \geq 2$, then $\det M = \prod_{1\leq i<j\leq n}(a_j - a_i)$.

Question 4. Suppose $p(x) \in \mathbb{F}[x]$ is a polynomial, and $T : V \to V$ is a linear transformation on a (not necessarily finite-dimensional) $\mathbb{F}$-vector space V.

(1) If T has an eigenvalue λ, then prove that the linear transformation $p(T) : V \to V$ has an eigenvalue $p(\lambda)$.

(2) More generally, let $c_i, \lambda_i \in \mathbb{F}$, $v_i \in V$, and $Tv_i = \lambda_i v_i$ for $1 \leq i \leq k$. Prove (as asserted in class) that

$$p(T) \sum_{i=1}^{k} c_i v_i = \sum_{i=1}^{k} c_i p(\lambda_i) v_i.$$

Question 5. Suppose $\mathbb{F} = \mathbb{Z}/5\mathbb{Z} = \mathbb{F}_5$, and $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Compute the eigenvalues of A and the λ-eigenspace for every scalar λ.

Question 6. The *Fibonacci numbers* are defined recursively/inductively as:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1} \ \forall n \geq 1.$$

Every number is the sum of the previous two terms: $0, 1, 1, 2, 3, 5, 8, \ldots$

The goal of this exercise is to *derive* the following closed-form expression for f_n, termed *Binet's formula*:

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

(Certainly once the formula is known, it is easy to prove it by induction. But how does one obtain this formula in the first place?)

(1) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Show that $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$ for all $n \geq 0$.
(2) Find the eigenvalues and a choice of eigenvectors of A, each of which has *unit length* (as a vector in $\mathbb{R}^2$).
(3) Using this, write $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P (if you have done things right, you should get that $PP^T = \mathrm{Id}$, so that $P^{-1} = P^T$). The entries of D should be $(1 \pm \sqrt{5})/2$.
(4) Finally, compute f_n.

Question 7. If $p(x) \in \mathbb{F}[x]$, and $A \in \mathbb{F}^{n \times n}$ is a block-triangular matrix of the form

$$\begin{pmatrix} B_{k \times k} & C_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & D \end{pmatrix},$$

then show that $p(A) = \begin{pmatrix} p(B) & C' \\ \mathbf{0} & p(D) \end{pmatrix}$ for some matrix C'.