MA219 – Linear Algebra
2024 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 9 (*due by Monday, November 11* in TA's office)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Compute $A^3 + A^2 + A$, without multiplying 3×3 matrices. (Hint: Compute the characteristic polynomial of A.)

Question 2. (This is related to the "long proof" that we saw on Tuesday October 29, about a matrix being diagonalizable if and only if its minimal polynomial has no repeated roots.) Suppose $p(x) \in \mathbb{F}[x]$ is any polynomial of degree $d > 0$. Show that p has at most d distinct roots in $\mathbb{F}$. As a hint: use Question 3 from HW7 about when a Vandermonde matrix is invertible.

Question 3. Suppose $\lambda \in \mathbb{F} = \mathbb{R}$ and $J = J(3, \lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ is a Jordan block.

(1) Write down a formula for J^k for any integer $k \geq 1$, and prove it.
(2) More generally, if f is a polynomial with real coefficients, prove that

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & f''(\lambda)/2! \\ 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & f(\lambda) \end{pmatrix}.$$

(3) Write down (but don't prove) a formula for $f(J)$, where f is an arbitrary polynomial with real coefficients, and $J = J(n, \lambda)$ for arbitrary $n \geq 1$. As above, the Jordan block $J(n, \lambda)$ is the $n \times n$ upper triangular matrix, with λ on the diagonal and 1 on the super-diagonal (and all other entries zero).

Question 4. Suppose $\mathbb{F}$ is any field, $\lambda \in \mathbb{F}$ is any scalar, and $n \geq 1$ is any integer. Let $J = J(n, \lambda)$ be a Jordan block.

(1) Compute the algebraic and geometric multiplicities of all eigenvalues of J.

(2) Show that the minimal and characteristic polynomials of J agree.
(3) Compute the kth power of $J(n, 0)$, for all integers $k \geq 1$.

Question 5. Suppose a real matrix A can be written in Jordan canonical form, with Jordan blocks

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \qquad (4), \qquad (1), \qquad (0), \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Compute the following, with some reasoning.
(1) The characteristic polynomial of A.
(2) The minimal polynomial of A.
(3) The algebraic and geometric multiplicities of all eigenvalues of A.
(4) The rank of A.

Question 6. This question shows that every *complex* square matrix is conjugate to its transpose. (The same holds true over every field, but this is harder.)
(1) Show that a Jordan block matrix over any field, say $J = J(n, \lambda) \in \mathbb{F}^{n \times n}$, is conjugate to its transpose: $J^T = PJP$, where $P = P^{-1} = P^T$ is the matrix with 1s along the *anti-diagonal*. In other words, $P_{ij} = 1$ if $j = n + 1 - i$, and 0 otherwise.
(2) Now suppose $A \in \mathbb{C}^{n \times n}$. Show that $A^T = QAQ^{-1}$ for some $Q \in \mathbb{C}^{n \times n}$ invertible.