MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Friday, August 22* in TA's office hours, or previously in class)

Question 1. Fix a prime $p \geq 2$ and let a, b, c, d be integers. If $a \equiv b \mod p$ and $c \equiv d \mod p$, show that $ac \equiv bd \mod p$.

Question 2. Given a group G and an element $g \in G$, let $L_g : G \to G$ denote the *left-translation* map, sending any element h to gh. Show that L_g is a bijection on G.

Question 3. The goal here is to show that the set of complex numbers

$$\mathbb{C} := \{a + bi = a + b\sqrt{-1} : a, b \in \mathbb{R}\}$$

under the operations

$$(a + bi) + (c + di) := (a + c) + (b + d)i, \quad (a + bi) \cdot (c + di) := (ac - bd) + (ad + bc)i,$$
$$0 := 0 + 0i, \quad 1 := 1 + 0i,$$
$$-(a + bi) := (-a) + (-b)i, \quad (a + bi)^{-1} := \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$

is a field. (You are allowed to use that $\mathbb{R}$ is a field.)

I felt you should do some of these verifications at least once in your life – and it is not likely you will get to do this in any other course – so here, **prove that**:

(1) Multiplication is associative.
(2) $z \cdot z^{-1} = 1$ for all nonzero complex z.
(3) The distributive law holds.

Question 4. Suppose $\mathbb{F}$ is a field, with $a, b \in \mathbb{F}$. Prove the following statements.

(1) $(-1)^2 = 1$.
(2) $ab = 0$ in $\mathbb{F}$, if and only if $a = 0$ or $b = 0$.

Question 5. If $T : V \to W$ is $\mathbb{F}$-linear (as above), show that $T(\mathbf{0}_V) = \mathbf{0}_W$ and $T(-v) = -T(v)$ for all $v \in V$.

Question 6. Prove that a function $T : V \to W$ between two $\mathbb{F}$-vector spaces is $\mathbb{F}$-linear if and only if $T(cv + v') = cT(v) + T(v')$ for all $v, v' \in V$ and $c \in \mathbb{F}$.

Question 7. Fix a field $\mathbb{F}$, integers $m, n, p, q \geq 1$, and matrices $A_{m \times n}, B_{n \times p}, C_{p \times q}$ with entries in $\mathbb{F}$. Prove that:

(1) $A \cdot \mathrm{Id}_n = A$ – by checking the (i,k) (or (i,j)) entry on both sides.
(2) $A(BC) = (AB)C$ – by checking the (i,l) entry on both sides.
(3) $(AB)^T = B^T A^T$ for all $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times p}$ – again by checking entry by entry.
(4) $(B+C)A = BA+CA$ for $B, C \in \mathbb{F}^{m \times n}, A \in \mathbb{F}^{n \times p}$ – *without* using any entry-by-entry calculations, but using the previous part and that $A(B+C) = AB+AC$ (stated in class) whenever defined.
(5) Prove that the product of two lower triangular matrices is lower triangular. You are allowed to assume the corresponding statement for upper triangular matrices (which was stated in class).