MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 10 (*due by Friday, November 7* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. This exercise shows how to solve systems of first-order linear (ordinary) differential equations such as

$$x'(t) = 2x(t) + 3y(t), \qquad y'(t) = 3x(t) + 2y(t).$$

More generally, we work over $\mathbb{F} = \mathbb{R}$ and with a fixed integer $n \geq 1$ number of differentiable functions $x_1(t), \dots, x_n(t)$. Also fix a matrix $A = PDP^{-1}$ that is *diagonalizable*, i.e., P is invertible and D is diagonal, say with (i, i)-entry $\lambda_i \in \mathbb{R}$.

(1) First solve the system $\mathbf{x}'(t) = D\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$. (Use the initial values $x_i(0)$ too.)
(2) Now solve the system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Question 2. Suppose $A, B \in \mathbb{F}^{n\times n}$. We have seen that if A, B are similar/conjugate, then $p_A(x) = p_B(x)$. Is the converse true? Prove or give a counterexample. (E.g., do this for 2×2 matrices.)

Question 3. Suppose $A \in \mathbb{F}^{n\times n}$ has characteristic polynomial $p_A(x) = \det(x\mathrm{Id}_n - A)$. Write

$$p_A(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0.$$

(1) Explain why $c_n = 1$, $-c_{n-1}$ equals the trace of A, and $c_0 = (-1)^n \det A$.
(2) Suppose $\mathbb{F}$ contains n roots of the characteristic polynomial $p_A(x)$ (e.g., if it is an algebraically closed field). Prove that the sum and the product of the eigenvalues of A equal the trace and determinant of A, respectively.

Question 4. Suppose $T : V \to V$ is linear, with $\dim V = n \geq 1$. If T^k is the zero transformation for some integer $k \geq 1$, then show that $T^n = 0$. (Hint: If $k \geq n$, consider the minimal polynomial of T.)

Question 5. Suppose $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Compute $A^3 + A^2 + A$, without multiplying 3×3 matrices. (Hint: Compute the characteristic polynomial of A.)

Question 6. (This is related to the "long proof" that we saw on Tuesday October 29, about a matrix being diagonalizable if and only if its minimal polynomial has no repeated roots.) Suppose $p(x) \in \mathbb{F}[x]$ is any polynomial of degree $d > 0$. Show that p has at most d distinct roots in $\mathbb{F}$. As a hint: use a previous homework question about when a Vandermonde matrix is invertible.