MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 11 (*due by Friday, November 14* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

First see this video for about 14 minutes – for the definition of a Jordan block $J(n, \lambda)$ and the statement of the Jordan Canonical/Normal Form theorem: Lecture 23 starting from 36m45s to 50m. (The timestamp is in the hyperlink, so just click on it.)

Now work out the following questions:

Question 1. Suppose $\lambda \in \mathbb{F} = \mathbb{R}$ and $J = J(3, \lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ is a Jordan block.

(1) Write down a formula for J^k for any integer $k \geq 1$, and prove it.
(2) More generally, if f is a polynomial with real coefficients, prove that

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & f''(\lambda)/2! \\ 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & f(\lambda) \end{pmatrix}.$$

(3) Write down (but don't prove) a formula for $f(J)$, where f is an arbitrary polynomial with real coefficients, and $J = J(n, \lambda)$ for arbitrary $n \geq 1$. As above, the Jordan block $J(n, \lambda)$ is the $n \times n$ upper triangular matrix, with λ on the diagonal and 1 on the super-diagonal (and all other entries zero).
(4) As a special case, write down the kth power of $J(n, 0)$, for all integers $k \geq 1$.

Question 2. Suppose $\mathbb{F}$ is any field, $\lambda \in \mathbb{F}$ is any scalar, and $n \geq 1$ is any integer. Let $J = J(n, \lambda)$ be a Jordan block.

(1) Compute the algebraic and geometric multiplicities of all eigenvalues of J.
(2) Show that the minimal and characteristic polynomials of J agree. (Hint: use the previous question.)

Question 3. This question shows that every *complex* square matrix is conjugate to its transpose. (The same holds true over every field, but this is harder.)

(1) Show that a Jordan block matrix over any field, say $J = J(n, \lambda) \in \mathbb{F}^{n\times n}$, is conjugate to its transpose: $J^T = PJP$, where $P = P^{-1} = P^T$ is the matrix with 1s along the *anti-diagonal*. In other words, $P_{ij} = 1$ if $j = n + 1 - i$, and 0 otherwise.

(2) Now suppose $A \in \mathbb{C}^{n\times n}$. Show that $A^T = QAQ^{-1}$ for some $Q \in \mathbb{C}^{n\times n}$ invertible.