MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Friday, August 29* in TA's office hours, or previously in class)

Question 1. Given any field $\mathbb{F}$, prove the following formula for its prime subfield:

$$\mathbb{F}_{\text{prime}} = \bigcap_{\mathbb{K} \subseteq \mathbb{F} \text{ subfield}} \mathbb{K}.$$

Question 2. The *trace* of a square matrix $A = (a_{ij})_{i,j=1}^n$ is the sum of its diagonal entries: $a_{11} + a_{22} + \cdots + a_{nn}$. Given integers $m, n \geq 1$ and matrices $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times m}$, prove that AB and BA have the same trace, even if they have different sizes.

Question 3. Suppose $A \in \mathbb{F}^{m \times n}, B \in \mathbb{F}^{n \times p}$ are invertible. Prove that A^T and AB are invertible, by showing that $(A^T)^{-1} = (A^{-1})^T$ and $(AB)^{-1} = B^{-1}A^{-1}$. (Prove only from one side, e.g. $M \cdot M^{-1} = \mathrm{Id}$, not from both sides.)

Question 4. (a) Solve the following system:

$$2x + 3y + z = 1, \quad x - y - z = 2$$

over a field of characteristic $\neq 5$.

(b) Solve the following system:

$$x + y + z = 1, \quad x + 2y + 3z = 2, \quad x + 4y + 9z = 7$$

over a field of characteristic $\neq 2$.

Question 5. In class, we saw that the set of functions from any set to a field is a vector space. Now let us see why **every** vector space is a subspace of such a space of functions.

Let B be a nonempty set, and $\mathbb{F}$ a field. Define $Fun_0(B, \mathbb{F})$ to be the set of all functions $f : B \to \mathbb{F}$ such that $f(b) = 0$ for all but finitely many $b \in B$. Show that $Fun_0(B, \mathbb{F})$ is an $\mathbb{F}$-vector space.

Note: You are allowed to use the fact mentioned in class, that the set of *all* functions $Fun(B, \mathbb{F})$ is an $\mathbb{F}$-vector space, which might help bypass a lot of the routine verifications for the subset Fun_0.

(Later, we will say that every $\mathbb{F}$-vector space V has a "basis" B, and then the space V can in fact be identified precisely with our space here: $Fun_0(B, \mathbb{F})$.)