MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 3 (*due by Friday, September 5* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $\mathbb{F}$ is a field, and $n \geq 1$ an integer. For integers $1 \leq i, j \leq n$, define the $n \times n$ matrix E_{ij} as having all entries zero, except 1 in the (i, j)-entry.

Now find the $\mathbb{F}$-span of the following sets – give (with some justification – maybe via the explicit description) the "conceptual description" (see e.g. towards the end of Lecture L06 in the videos).

(1) The matrices E_{ii} for $1 \leq i \leq n$.
(2) The polynomials $x^2 - x, x^3 - x^2, \ldots$ and the polynomial x, with $\mathbb{F} = \mathbb{R}$.

Question 2. For each of the following, explain whether or not the specified subset (of the corresponding vector space) is a subspace.

(1) The subset of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying: $f(1) - f(2) + 4f(3) = 0$.
(2) The subset of solutions to $Ax = b$ for some vector $b \neq 0$. Here $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ for some integers $m, n \geq 1$.

Question 3. Suppose V is a vector space over $\mathbb{F}$.

(1) Show that a nonempty subset W is a subspace if and only if for all $w, w' \in W$ and scalars $c \in \mathbb{F}$, the vector $cw + w' \in W$.
(2) Suppose $L \subset V$ is a linearly independent subset. If $v \in V$ is not in the span of L, show that $L \cup \{v\}$ is also linearly independent.

Question 4.

(1) Show that three vectors in $\mathbb{R}^2$ are linearly dependent.
(2) Find three vectors in $\mathbb{R}^2$ such that any two are linearly independent.