MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Friday, September 12* in the TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose V, W are $\mathbb{F}$-vector spaces, and $T : V \to W$ is an $\mathbb{F}$-linear transformation. If T is a bijection (i.e. an isomorphism), show that the inverse map T^{-1} is also a linear transformation.

Question 2. Suppose V, W are $\mathbb{F}$-vector spaces. Show that $Lin_{\mathbb{F}}(V, W)$, the space of $\mathbb{F}$-linear maps : $V \to W$, is a vector subspace of $Fun(V, W)$. (You can assume that the latter is an $\mathbb{F}$-vector space.)

Question 3. Suppose $A, B \in \mathbb{F}^{m\times n}$ for some integers $m, n \geq 1$. Prove that the following are equivalent:

(1) $A = B$.
(2) $Av = Bv$ for all vectors $v \in \mathbb{F}^n$.
(3) $A\mathbf{e}_j = B\mathbf{e}_j$ for all $1 \leq j \leq n$.

Question 4. Let $\mathbb{F} = \mathbb{Q}$.

(1) Show that $\mathbb{R}$ is not a finite-dimensional $\mathbb{Q}$-vector space.
(2) Suppose V is a *countable*-dimensional $\mathbb{Q}$-vector space, i.e. a vector space with a countably infinite basis $v_1, v_2, \ldots, v_n, \ldots$ Show that V is the union of its finite-dimensional subspaces V_n spanned by $v_1, \ldots, v_n$.
(3) Show that $\mathbb{R}$ is not a countable-dimensional $\mathbb{Q}$-vector space.

Question 5.

(1) Suppose S_1, S_2 are subsets of an $\mathbb{F}$-vector space. Prove that S_1, S_2 have the same spans if and only if each set is contained in the span of the other.
(2) The *row space* of a matrix is the span of its rows. If $A, B \in \mathbb{F}^{m\times n}$ are row-equivalent, prove that their row spaces are equal.

Question 6. Suppose S is a linearly independent subset of a vector space W (over a field $\mathbb{F}$). Consider a chain of linearly independent subsets in W:

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots$$

Prove that $\bigcup_{i\geq 0} S_i$ is also a linearly independent subset. (In a special case, this is the 'upper bound' of a 'chain' that is used in proving that every vector space has a basis, via Zorn's Lemma.)