MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 5 (*due by Friday, September 19* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $\mathbb{F}$ is a finite field of size $q \geq 2$, and V is an $\mathbb{F}$-vector space.

(1) If V is an $\mathbb{F}$-vector space, show that V is not the union of q-many proper subspaces. Hence V is not the union of n proper subspaces, for any $0 \leq n \leq q$. (The proof also shows that e.g. $\mathbb{R}^k$ is not the union of finitely many proper subspaces, for any $k \geq 1$.)

*(Hint, for one possible approach: Suppose V **is** the union of q proper subspaces – let $2 \leq m \leq q$ be the smallest number of subspaces needed to cover V, say $W_1, \dots, W_m \subset V$. Then there exist $w_i \in W_i$ such that $w_i \notin W_j$ for all $j \neq i$. Now consider certain $(q+1)$-many linear combinations of w_1, w_2.)*

(2) In this part and the next, we will show that if we instead had $n \geq q+1$ (in fact $n = q+1$), then V can be a union of n proper subspaces. To see why, first show here that $\mathbb{F}^2$ is a union of $q+1$ proper subspaces.

(3) Now suppose $V \neq 0$ is an arbitrary $\mathbb{F}$-vector space of dimension at least 2 (and possibly infinite), and B is a basis of V. (You may assume B exists.) Show that V is a union of $q+1$ proper subspaces.

Question 2. Suppose V is an $\mathbb{F}$-vector space, with ordered basis $\mathcal{B} = (v_1, \dots, v_n)$. Prove that the map $\eta : V \to \mathbb{F}^n$, sending a vector $v = c_1 v_1 + \cdots + c_n v_n$ to the column vector $[v]_{\mathcal{B}} = (c_1, \dots, c_n)^T$, is a vector space isomorphism.

Question 3. Suppose $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $\theta \in \mathbb{R}$. Suppose $T : V \to V$ is the linear transformation that rotates a vector counterclockwise by θ (radians). Compute the matrix of T with respect to the standard basis of V.

Question 4. Suppose $\mathbb{F}$ is a field, and $T : \mathbb{F}^2 \to \mathbb{F}^2$ is the linear operator $T(x_1, x_2) := (x_2, -x_1)$, where $(x_1, x_2)^T = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$ is with respect to the standard ordered basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$.

(1) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}}$?
(2) What is the matrix of T given by $[T]_{\mathcal{B},\mathcal{B}'}$, where $\mathcal{B}' = (\mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1)$?
(3) What is the transition matrix of $\mathcal{B}'$ into $\mathcal{B}$? Meaning, find the matrix P such that $[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$ for all $v \in \mathbb{F}^2$.
(4) Suppose $\mathbb{F}$ has characteristic not 2 (so $2 = 1+1$ in $\mathbb{F}$). What is the coordinate vector of $(-1, 2)^T$ in the standard basis, when written out in the basis $\mathcal{B}'$?