MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 6 (*due by Friday, October 3* in TA's office hours)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. We have seen that (by convention,) the zero vector space over any field has exactly one basis: the empty set. Now:

(1) Classify all nonzero vector spaces over all fields, which also have exactly one unordered basis – i.e., exactly one basis up to permuting its basis elements.

(2) Classify all nonzero vector spaces over all fields, which have exactly two unordered bases.

Question 2. Suppose $\mathbb{F}$-vector spaces V, W, X have ordered bases $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_m)$, and $(x_1, \ldots, x_p)$ for positive integers n, m, p respectively. Write down a basis of the vector space of bilinear maps : $V \times W \to X$, and prove that it is a basis.

Question 3. Recall that <u>one candidate</u> for the *coproduct* of a family $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces is their direct sum, denoted

$$\coprod_{i \in I} V_i = \bigoplus_{i \in I} V_i,$$

with a typical element $(v_i)_{i \in I}$ where all but finitely many v_i are zero. This vector space comes with fixed *injection* maps

$$\iota_j : V_j \to \coprod_{i \in I} V_i, \qquad v_j \mapsto (v_i)_{i \in I},$$

where $v_i = v_j$ if $i = j$ and $v_i = \mathbf{0}_{V_i}$ otherwise.

(1) We now verify that the coproduct conditions (and more) are satisfied. Start by verifying that each ι_j is an injective $\mathbb{F}$-linear map.

(2) Write out the (complete) proof that this product satisfies the following "universal property":

> *Given any $\mathbb{F}$-vector space Z, and $\mathbb{F}$-linear maps $\varphi_j : V_j \to Z$ for all $j \in I$, there exists a unique $\mathbb{F}$-linear map $\varphi : \oplus_{j \in I} V_j \to Z$ such that $\varphi_j = \varphi \circ \iota_j$ for all $j \in I$.*

In other words, the direct sum yields the existence of an object that satisfies this universal property. (By class, every other "candidate" is isomorphic to this one.)

Question 4. Here we see the difference between linear maps and bilinear maps – which is at the heart of why the coproduct (= direct sum / Cartesian product) and tensor product of two vector spaces have different dimensions.

Suppose V, W are two finite-dimensional vector spaces over $\mathbb{F}$, with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ respectively.

(1) Prove that $\mathrm{Lin}(V \times W, \mathbb{F})$ has a basis given by the $n + m$ linear maps, which take one of the elements $\{v_i \oplus 0, 0 \oplus w_j\}$ to 1 and the others to zero (and extending by linearity).

(2) Prove that the $n \cdot m$ maps indexed by $i_0 \in [1, n]$ and $j_0 \in [1, m]$, given by

$$\varphi_{i_0,j_0}\left(\sum_{i=1}^{n} c_i v_i, \sum_{j=1}^{m} d_j v_j\right) := c_{i_0} d_{j_0}$$

form an $\mathbb{F}$-basis of $\mathrm{BiLin}(V \times W, \mathbb{F})$.

Question 5. (*Quotient spaces – you don't need to submit this solution.*) Suppose V is an $\mathbb{F}$-vector space, and $W \subset V$ a subspace. Define a relation $v \sim v'$ on V if $v - v' \in W$. Now define the quotient space V/W to be the set of distinct equivalence classes $[v]$, under the relation:

$$a[v] + b[v'] := [av + bv'], \qquad a, b \in \mathbb{F},\ v, v' \in V.$$

(1) Verify that $\sim$ above is an equivalence relation on V.

(2) Check that the addition defined above (setting $a = b = 1$) is associative.

(3) Check that $\mathbf{0}_{V/W} := [w]$ for $w \in W$ is an (or 'the') additive identity in V/W.

(4) Assuming that V/W is a vector space under the above operations (which it is!), check that the map $\pi : V \to V/W$ sending $v \in V$ to $[v]$ is a surjective $\mathbb{F}$-linear map.

(5) Prove that the quotient space satisfies the following 'universal property':

Given any $\mathbb{F}$-vector space Z, and a $\mathbb{F}$-linear map $\varphi : V \to Z$ which maps W to $\mathbf{0}_Z$, there exists a unique $\mathbb{F}$-linear map $\overline{\varphi} : V/W \to Z$ such that $\varphi = \overline{\varphi} \circ \pi$ (with π as in the previous part).

Question 6. Prove that $\mathbb{R}^2/\mathbb{R}\mathbf{e}_1 \cong \mathbb{R}\mathbf{e}_2$ as $\mathbb{R}$-vector spaces.