MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 7 (*due by Friday, October 17* in TA's office hours)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose an $\mathbb{F}$-vector space V has an ordered basis $(v_1, \dots, v_n)$. For $1 \leq i_0 \leq n$, define the *dual functionals* $\varphi_{i_0} : V \to \mathbb{F}$ via:

$$\varphi_{i_0}\left(\sum_{i=1}^{n} c_i v_i\right) := c_{i_0}.$$

Assuming that $\varphi_{i_0} \in V^*$, show that these vectors form a basis of V^*. (This is called the *dual basis.*)

Question 2. Suppose $V = \mathbb{F}^n$ is the space of column vectors, so that (as we saw in class) $V^* \cong (\mathbb{F}^n)^T$ is its dual space of row vectors.

(1) Now suppose $v_1, \dots, v_n \in \mathbb{F}^n$ are an n-tuple of n-vectors that constitute an ordered basis. Set $A := [v_1|v_2|\cdots|v_n]$. Compute the dual basis vectors (which are row vectors, or n-tuples) in terms of A.

(2) As a special case, suppose $n = 2$, $v_1 = (a, c)^T$, and $v_2 = (b, d)^T$. Compute the dual basis.

Question 3. Recall the direct product and direct sum (or coproduct) of a set $\{V_i : i \in I\}$ of $\mathbb{F}$-vector spaces, constructed in class (and studied in the preceding homework set). The goal of this exercise is to show that

$$\left(\bigoplus_{i\in I} V_i\right)^* \cong \prod_{i\in I} V_i^*,$$

by going the 'reverse' way:

(1) Given $\mathbf{\Phi} = (\varphi_i)_{i\in I}$, with each $\varphi_i \in V_i^*$, first show that $(\varphi_i)_{i\in I}$ yields a linear map from $\bigoplus_{i\in I} V_i$ to $\mathbb{F}$. Let us call this map $T(\mathbf{\Phi})$.

(2) Show that the assignment $T : \mathbf{\Phi} \mapsto T(\mathbf{\Phi})$ is a linear map, from $\prod_{i\in I} V_i^*$ to $\left(\bigoplus_{i\in I} V_i\right)^*$.

(3) Show that T is one-to-one and onto.

Question 4. Fix a field $\mathbb{F}$, an integer $n \geq 1$, and nonzero $\mathbb{F}$-vector spaces $V_1, \ldots, V_n, W$. Verify that the set of multilinear maps : $V_1 \times \cdots \times V_n \to W$ is a vector space.