MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 8 (*due by Friday, October 24* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Using results from class about how the determinant changes under elementary row operations (or other results about the determinant), compute the determinants of the matrices $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Question 2. Prove from "first principles" (and the main Theorem on determinants that we proved in class on Tuesday) that: the determinant of an upper triangular matrix equals the product of its diagonal entries.

Question 3. A *Vandermonde* matrix is a matrix of the form

$$M_{n\times n} = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

where $n \geq 1$ is an integer, and $a_1, \ldots, a_n \in \mathbb{F}$ are scalars.

Prove (e.g. by induction on n) that if $n \geq 2$, then $\det M = \prod_{1\leq i<j\leq n}(a_j - a_i)$.