MA219 – Linear Algebra
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 9 (*due by Friday, October 31* in TA's office hours, or previously in class)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Given a square matrix $A \in \mathbb{F}^{n\times n}$, define its *adjugate* matrix $adj(A) \in \mathbb{F}^{n\times n}$ to have (i,j) entry $(-1)^{i+j} \det A_{j|i}$, where $A_{j|i} \in \mathbb{F}^{(n-1)\times(n-1)}$ is the matrix obtained by removing the jth row and ith column of A. Prove the following properties for any matrix $A \in \mathbb{F}^{n\times n}$, say with $n \geq 2$:

(1) $adj(A) \cdot A = A \cdot adj(A) = (\det A)\mathrm{Id}_n$.
(2) If A is singular then $adj(A)$ is also singular.
(3) $\det(adj A) = (\det A)^{n-1}$.
(4) $adj(A^T) = adj(A)^T$.

Question 2. Suppose $p(x) \in \mathbb{F}[x]$ is a polynomial, and $T : V \to V$ is a linear transformation on a (not necessarily finite-dimensional) $\mathbb{F}$-vector space V.

(1) If T has an eigenvalue λ, then prove that the linear transformation $p(T) : V \to V$ has an eigenvalue $p(\lambda)$.
(2) More generally, let $c_i, \lambda_i \in \mathbb{F}$, $v_i \in V$, and $Tv_i = \lambda_i v_i$ for $1 \leq i \leq k$. Prove (as asserted in class) that

$$p(T)\sum_{i=1}^{k} c_i v_i = \sum_{i=1}^{k} c_i p(\lambda_i) v_i.$$

Question 3. Suppose $\mathbb{F} = \mathbb{Z}/5\mathbb{Z} = \mathbb{F}_5$, and $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Compute the eigenvalues of A and the λ-eigenspace for every scalar λ.

Question 4. The *Fibonacci numbers* are defined recursively/inductively as:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1} \ \forall n \geq 1.$$

Every number is the sum of the previous two terms: $0, 1, 1, 2, 3, 5, 8, \ldots$

The goal of this exercise is to *derive* the following closed-form expression for f_n, termed *Binet's formula*:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(Certainly once the formula is known, it is easy to prove it by induction. But how does one obtain this formula in the first place?)

(1) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Show that $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$ for all $n \geq 0$.

(2) Find the eigenvalues and a choice of eigenvectors of A, each of which has *unit length* (as a vector in $\mathbb{R}^2$).

(3) Using this, write $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P (if you have done things right, you should get that $PP^T = \mathrm{Id}$, so that $P^{-1} = P^T$). The entries of D should be $(1 \pm \sqrt{5})/2$.

(4) Finally, compute f_n.

Question 5. If $p(x) \in \mathbb{F}[x]$, and $A \in \mathbb{F}^{n\times n}$ is a block-triangular matrix of the form

$$\begin{pmatrix} B_{k\times k} & C_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & D \end{pmatrix},$$

then show that $p(A) = \begin{pmatrix} p(B) & C' \\ \mathbf{0} & p(D) \end{pmatrix}$ for some matrix C'.