MA315 – Lie Algebras and their Representations
2024 Autumn Semester

- You are expected to write (or present in class) proofs / arguments with reasoning, in solving these questions.
- Also, the convention "Exercise $m.n$" below refers to the nth Exercise at the end of Section m, in the course textbook by Humphreys.

Problem Set 1 – submit in-class on Thu Sep 5

Question 1. Prove that the Jacobi identity holds in any associative ring with the commutator bracket.

Question 2. Suppose L is an $\mathbb{F}$-vector space with a bilinear map $[\cdot,\cdot] : L \times L \to L$ such that $[x,x] = 0$ for all $x \in L$. Prove that the following are equivalent:

(1) L is a Lie algebra, i.e. the Jacobi identity holds.
(2) For all $x \in L$, the map $\operatorname{ad} x : L \to L$ given by $y \mapsto [x,y]$, is a derivation on L.
(3) The adjoint map $\operatorname{ad} : L \to \operatorname{End}_{\mathbb{F}}(L)$ is a Lie algebra homomorphism.

Question 3. Suppose $L = \mathfrak{sl}_2(\mathbb{F})$, with $\operatorname{char}(\mathbb{F}) \neq 2, 3$. As usual, let the nilpotent matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = x^T.$$

(1) Define $g := \exp(x)\exp(-y)\exp(x)$. Show that $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In particular, $g^2 = -\mathrm{Id}$, and so conjugation by g is an order two Lie algebra automorphism of L.

(2) Define $\sigma := \exp(\operatorname{ad} x)\exp(\operatorname{ad} -y)\exp(\operatorname{ad} x)$. Show that $\sigma \equiv \operatorname{Ad}_g$ (conjugation by g) on L, which sends $x \longleftrightarrow -y$ and sends h to $-h$.

Question 4. Prove that the normalizer of any Lie subalgebra of L, is itself a Lie subalgebra.

Question 5 (Exercise 2.1)**.** Prove that the set of inner derivations $\operatorname{ad} x$, $x \in L$ forms an ideal of $\operatorname{Der} L$.

Question 6 (Exercise 2.2)**.** Show that $\mathfrak{sl}_n(\mathbb{F})$ is the derived algebra $[L,L]$ of $L = \mathfrak{gl}_n(\mathbb{F})$, for any integer $n \geq 1$ and field $\mathbb{F}$.

Question 7 (Exercise 3.4)**.** Prove that L is solvable (resp. nilpotent) if and only if $\operatorname{ad} L$ is so.

Question 8 (Exercise 3.6)**.** Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal.

Question 9 (Exercise 4.3)**.** Here is an example of when/how Lie's theorem fails when $\mathbb{F}$ has nonzero characteristic, say $p > 0$. Consider the $p \times p$ matrices

$$x = \begin{pmatrix} \mathbf{0}_{(p-1)\times 1} & \mathrm{Id}_{p-1} \\ 1 & \mathbf{0}^T \end{pmatrix}, \quad y = \mathrm{diag}(0, 1, 2, \dots, p-1).$$

(1) Verify that $[x, y] = x$. Thus, their span $L \subseteq \mathfrak{gl}_p(\mathbb{F})$ is solvable.
(2) However, (verify that) x, y have no common (nonzero) eigenvector.

Question 10 (Exercise 4.5)**.** If $x, y \in \mathrm{End}(V)$ commute, prove that $(x+y)_s = x_s + y_s$ and $(x + y)_n = x_n + y_n$.

Question 11 (Exercise 4.6)**.** Check formula $(*)$ at the end of (4.2).

Question 12 (Exercise 4.7)**.** Prove the converse of Theorem 4.3 (Cartan's crierion).

Question 13 (Exercise 5.1)**.** Prove that if L is nilpotent, the Killing form of L is identically zero.

Question 14 (Exercise 1.6)**.** Let $x \in \mathfrak{gl}_n(\mathbb{F})$ have n distinct eigenvalues $a_1, \dots, a_n \in \mathbb{F}$. Show that the eigenvalues of $\mathrm{ad}\, x$ are precisely the n^2 scalars $a_i - a_j$ for $1 \le i, j \le n$, by identifying the eigenvectors / eigen-matrices (with respect to a suitable basis of $\mathbb{F}^n$).

<u>**Problem Set 2 – submit in-class on Tue Oct 8**</u>

Question 15 (Exercises 5.5 and 6.1)**.** Let $L = \mathfrak{sl}_2(\mathbb{F})$. Compute the basis of L dual to the standard basis (x, h, y), with respect to the Killing form.

Now compute the Casimir element with respect to these two dual bases.

Question 16 (Exercise 5.8)**.** Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are (respectively) the sums of the semisimple and nilpotent parts in the various L_i of the components of x.

Question 17. Suppose L is a *simple* Lie algebra and let $\phi : L \to \mathfrak{gl}(V)$ be a nonzero representation.

(1) Prove that ϕ is faithful.

(2) Now as explained at the start of Section 6.2, the bilinear form $\beta(x, y) := \mathrm{Tr}(\phi(x)\phi(y))$ on $L \times L$ is symmetric, associative, and nondegenerate. Now let $\{x_1, \dots, x_n\}$ be any basis of L, and $y_1, \dots, y_n$ the corresponding dual basis. Prove that the Casimir element $c_\phi(\beta) := \sum_{i=1}^n \phi(x_i)\phi(y_i)$ is independent of the choice of basis $\{x_i\}$.

Question 18 (Exercise 6.2)**.** Let L be a Lie algebra and V be an L-module. Show that V is a direct sum of irreducible submodules if and only if each L-submodule of V possesses a complement.

Question 19 (Exercise 6.6)**.** Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]

Question 20 (Exercise 6.7)**.** It will be seen later on that $\mathfrak{sl}_n(\mathbb{F})$ is actually *simple*. Assuming this and using the preceding exercise, prove that the Killing form κ on $\mathfrak{sl}_n(\mathbb{F})$ is related to the ordinary trace form by $\kappa(x, y) = 2n\,\mathrm{Tr}(xy)$.

Question 21 (Exercise 7.2)**.** $M = \mathfrak{sl}_3(\mathbb{F})$ contains a copy of $L = \mathfrak{sl}_2(\mathbb{F})$ in its upper left-hand 2×2 position. Write M as a direct sum of irreducible L-submodules (M viewed as L-module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Question 22 (Exercise 7.4, essentially)**.** A concrete example of the irreducible finite-dimensional $\mathfrak{sl}_2$-module $V(n)$ is provided by the space P_n of homogeneous polynomials in X, Y of total degree $n \geq 0$. That is, define

$$P_n := \ker\left(-n + X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}\right) \subset \mathbb{F}[X, Y].$$

Now define $\rho_n : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}_{\mathbb{F}}(P_n)$ via:

$$\rho_n(x) := X\frac{\partial}{\partial Y}, \qquad \rho_n(y) := Y\frac{\partial}{\partial X}, \qquad \rho_n(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Show that $P_n \cong V(n)$ as $\mathfrak{sl}_2(\mathbb{F})$-modules.

Question 23 (Exercise 7.7, essentially)**.** We now construct *infinite-dimensional* examples of modules similar to $V(n)$: the **Verma modules**. For this, fix a scalar λ and consider the polynomial algebra $M(\lambda) \cong \mathbb{F}[y]$. Here, $\mathfrak{sl}_2(\mathbb{F})$ acts by differential operators on $M(\lambda)$ – e.g., y maps to multiplying by y:

$$y \mapsto y\ , \qquad h \mapsto \lambda - 2y\partial_y, \qquad x \mapsto \lambda\partial_y - y\partial_y^2.$$

Verify that the $\mathfrak{sl}_2$-relations hold among these differential operators.

Question 24. Suppose V is a finite-dimensional $\mathfrak{sl}_2$-module. If the h-weights of V are of the form $\{k, k-2, \dots, -k\}$ for some non-negative integer k, and if every weight space ($= h$-eigenspace) has dimension 1, then show that V is irreducible.

Problem Set 3 – submit on Mon Nov 18

Question 25 (Exercise 8.5)**.** If L is semisimple and H is a maximal toral subalgebra of L, prove that H is self-normalizing: $H = N_L(H)$.

Question 26 (Exercise 9.2)**.** Prove that $\Phi^\vee$ is a root system in E if Φ is, and then their two Weyl groups are isomorphic. Also check moreover that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in \Phi$.

Question 27 (Exercise 9.6)**.** Prove that W is a normal subgroup of $\mathrm{Aut}(\Phi)$.

Question 28 (Exercise 10.1)**.** Prove that in the dual root system $\Phi^\vee$ to Φ, for any base $\Delta \subseteq \Phi$ the dual subset $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$ is a base of $\Phi^\vee$. [Hint: Compare the Weyl chambers of Φ and $\Phi^\vee$.]

Question 29 (Exercise 10.5)**.** If $\sigma \in W$ can be written as a product of t simple reflections, prove that t has the same parity as $\ell(\sigma)$.

Question 30 (Exercise 10.13)**.** Show that the only reflections in W are of the form σ_α for $\alpha \in \Phi$. [Hint: A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in W.]

Question 31 (Exercise 11.4)**.** Prove that the Weyl group of a root system is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

Question 32 (Exercise 13.7)**.** If $\varepsilon_1, \dots, \varepsilon_l$ is an *obtuse* basis of the Euclidean space E (i.e., all inner products $(\varepsilon_i, \varepsilon_j) \leq 0$ for $i \neq j$, then prove that the dual basis is *acute* – i.e., all of their inner products are non-negative. [Reduce to the case $l = 2$.]

Question 33 (Exercise 13.9)**.** Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda + \delta) - \delta$ is dominant if and only if $\sigma = 1$.

Question 34. Prove that the Cartan matrix M of a semisimple Lie algebra is *symmetrizable*, i.e. there exists a diagonal matrix D and a real symmetric matrix B such that $A = DB$.