MA341 – Matrix Analysis and Positivity
2025 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 1 (*due by Thursday, September 11* in class, or previously in office hours)

Question 1 (*The correlation trick*)**.** Recall that a positive semidefinite matrix is a *correlation matrix* if all its diagonal entries are 1.

(1) Prove that for every positive definite matrix A, there exists a unique positive definite diagonal matrix D and correlation matrix C such that $A = DCD$.
(2) Fix a dimension $n \geq 1$. Does the procedure in the previous part recover all $n \times n$ correlation matrices? Prove or find a counterexample.
(3) Prove that A and C have the same *pattern of zeros* and the same rank. By the former, we mean that if $a_{jk} = 0$ for some j, k then $c_{jk} = 0$ as well.

Question 2. Suppose $n \geq 1$ is an integer and $C_{n\times n}$ is a correlation matrix, i.e. C is positive semidefinite with all diagonal entries 1.

(1) Show that $n\mathrm{Id} - C$ is positive semidefinite.
(2) Show with an example that the coefficient n in the preceding question is sharp (i.e., cannot be reduced).
(3) More generally, show that if $A \in \mathbb{P}_n$ and D is the diagonal matrix with (j, k)-entry $\delta_{j,k} a_{jj}$, then $nD - A$ is positive semidefinite.

Question 3. If the columns of an $m \times n$ real matrix A are linearly independent, verify that its Moore–Penrose inverse is $A^\dagger = (A^T A)^{-1} A^T$ (including showing that $A^T A$ is invertible).

Question 4. We now discuss a different partition of LPM-matrices: via their *negative inertia*, which is the number of negative eigenvalues. (Recall that the matrices are nonsingular, so all other eigenvalues are positive.)

(1) Fix an integer $n \geq 1$ and a sign pattern $\vec{\epsilon} \in \{\pm 1\}^n$. Show that every matrix in $LPM_n(\vec{\epsilon})$ has the same negative inertia, and compute this integer in terms of $\vec{\epsilon}$.

(2) "Dually", given an integer $0 \leq k \leq n$, count – with reasoning – the number of $\vec{\epsilon}$ such that $\mathbb{D}_{\vec{\epsilon}}$ (and hence $LPM_n(\vec{\epsilon})$, by the previous part) has negative inertia k. (Hint: a corollary is the "first" special case of the binomial theorem!)

Question 5. Fix an integer $n \geq 1$, sign patterns $\vec{\epsilon}, \vec{\delta} \in \{\pm 1\}^n$, and a matrix $B_{\vec{\epsilon}} \in LPM_n(\vec{\epsilon})$. Write $B_{\vec{\epsilon}} = L_\circ \mathbb{D}_{\vec{\epsilon}} L_\circ^T$ using the generalized Cholesky factorization, and define the diagonal entries $k_{jj} := (L_\circ)_{jj}$.

(1) Perform the change-of-variables $L' := LL_\circ$. Show that the Jacobian of this square matrix of size $\binom{n+1}{2}$ is *upper* triangular! Compute its determinant in terms of the k_{jj}.

(2) Verify for every $L \in \mathbf{L}_n$ (the Cholesky space of lower triangular real matrices with positive diagonals) that $LB_{\vec{\epsilon}}L^T = L'\mathbb{D}_{\vec{\epsilon}}(L')^T$.

(3) Now show that there exists a "natural" matrix $B_{\vec{\delta}} \in LPM_n(\vec{\delta})$ such that $\Phi_{B_{\vec{\delta}}} \circ \Phi_{B_{\vec{\epsilon}}}^{-1} \equiv \Phi_{\mathbb{D}_{\vec{\delta}}} \circ \Phi_{\mathbb{D}_{\vec{\epsilon}}}^{-1}$ on all of $LPM_n(\vec{\epsilon})$. (In particular, the triangularity of the Jacobian and the unimodularity of its determinant follow.)

Question 6. Let $P_n \in \mathbb{R}^{n \times n}$ to be the anti-diagonal permutation matrix with (u, v) entry 1 if $u + v = n + 1$, and 0 otherwise. Now let A be a real symmetric matrix.

(1) Show that A is positive (semi)definite if and only if $P_n A P_n$ is so.

(2) Show yet another characterization of positive *definite* matrices: A (as above) is positive definite if and only if its *trailing* principal minors are positive.