MA341 – Matrix Analysis and Positivity
2023 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Thursday, October 30* in class, or previously in office hours)

Question 1 (*The (de)compression trick*)**.**

(1) Fix real scalars a, b, c and integers $m, n > 0$. Show that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive semidefinite (psd), if and only if the matrix $\begin{pmatrix} a & a & b \\ a & a & b \\ b & b & c \end{pmatrix}$ is psd,
or more generally, if and only if the block matrix $\begin{pmatrix} a\mathbf{1}_{n\times n} & b\mathbf{1}_{n\times m} \\ b\mathbf{1}_{m\times n} & c\mathbf{1}_{m\times m} \end{pmatrix}$ is psd.
Here, $\mathbf{1}_{m\times n}$ denotes the $m \times n$ matrix of all ones.

(2) (Merely) State the generalization of this result to arbitrary real symmetric matrices being positive semidefinite or not.

Question 2 (*Graph Laplacians*)**.** Suppose G is a weighted graph on nodes $1, \dots, n$. In other words, attach a non-negative real weight $w_{ij} = w_{ji}$ to each pair of nodes $\{i, j\}$ with $i \neq j$ (where $w_{ij} = 0$ denotes a lack of an edge). Now define the *graph Laplacian* of G to be the $n \times n$ matrix L_G with (i, j) entry $-w_{ij}$ for $i \neq j$, and (i, i) entry $\sum_{j\neq i} w_{ij}$.

Show that L_G is always positive semi-definite. (Try the 2×2 case first.)

Question 3 (Minimum matrices)**.**

(1) Suppose $x_1, \dots, x_n$ are nonnegative real numbers. Show that the matrix with (j, k)-entry $\min(x_j, x_k)$ is positive semidefinite. (Work this out in either of two ways: (a) write the matrix as a sum of rank-1 constant-entry-padded matrices; or (b) take Schur complements and use the induction hypothesis.)

(2) Show that if $0 < x_1 < x_2 < \cdots < x_n$, then the matrix in the preceding part is positive definite, with determinant $x_1 \prod_{j\geq 1}(x_{j+1} - x_j)$.

(3) Show next that if $m_1, \dots, m_n$ are nonnegative integers, and $p \geq 2$ is a prime integer, then the matrix with entries $p^{\min(m_j, m_k)}$ is positive semidefinite.

(4) Finally, if $l_1, \dots, l_n \geq 1$ are positive integers, then prove that their gcd matrix – i.e., the matrix with (j, k) entry $\gcd(l_j, l_k)$ – is positive semidefinite.

Question 4. Let $d \geq 0$ and let

$$A = \begin{pmatrix} p_1 & \alpha_2 & \cdots & \alpha_{d+1} \\ \alpha_2 & p_2 & & 0 \\ \vdots & & \ddots & \\ \alpha_{d+1} & 0 & & p_{d+1} \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}$$

be a real symmetric matrix.

(1) Show that $\det A = \prod_{j=1}^{d+1} p_j - \sum_{j>1} \alpha_j^2 \prod_{k=2,\ k\neq j}^{d+1} p_k$. (Hint: First do this for all $p_j \neq 0$, then extend by continuity to all p_j since the determinant is a polynomial function in the p_j, hence continuous.)

(2) Suppose $p_2, p_3, \cdots > 0$. Show that A is positive semidefinite if and only if $\det(A) \geq 0$.

Question 5. Another construction of new positive definite matrices from older ones: W. Pusz and S. L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys. 8 (1975), 159–170.

(1) Verify that the *geometric mean* of two positive definite (real) $n \times n$ matrices, given by

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

is positive definite.

(2) Verify that $A\#B$ is the unique positive definite solution X to the *Riccati equation* $XA^{-1}X = B$. (Hint: First do this for $A = \mathrm{Id}$.)

(3) Consequently, show that $A\#B = B\#A$, and

$$(C^{-1}AC)\#(C^{-1}BC) = C^{-1}(A\#B)C$$

for positive definite A, B and unitary C.

(4) When A, B commute, show that $A\#B = (AB)^{1/2}$.

Question 6. Every finite simple connected graph $G = (V, E)$ can be thought of as a metric space, by setting each edge to have unit length and assigning the distance between two nodes to be the length of the shortest path joining them. This question proves that the only graphs that isometrically embed into Hilbert space ℓ^2 are path graphs and complete graphs.

(1) Show that if $|V| \leq 3$, then G embeds isometrically into Hilbert space ℓ^2.

(2) Show that if $|V| = 4$, then G embeds isometrically into ℓ^2 if and only if G is either the path graph or the complete graph.

(3) Show that the only cycle that embeds isometrically into ℓ^2 is $C_3 = K_3$.

(4) Now suppose G is neither a path nor a cycle. Then G has a node v_0 of degree at least 3. (Why?) Assuming G embeds isometrically into ℓ^2, show that (a) v_0

is simplicial, i.e., its neighbors in G are all adjacent to each other. Now show that (b) G is complete.

(5) Finally, show that path graphs and complete graphs do indeed embed isometrically into Hilbert space.

Question 7. Suppose (X, d) is a metric space, and $z \in X$ is a fixed basepoint.

(1) Prove that the Kuratowski embedding $\Psi : X \to Fun(X, \mathbb{R})$ (the real-valued functions on X), given by

$$\Psi(x)(y) := d(x, y) - d(z, y), \qquad y \in X$$

is an isometric embedding of X into $C_b(X)$, the normed linear space of continuous bounded real-valued functions on X equipped with the sup-norm $\|\cdot\|_\infty$.

(2) Let $|X| = n + 1$. Then the recipe in the preceding part provides an isometric embedding into $\mathbb{R}^{n+1}$ with the sup-norm. Fréchet improved this to an embedding into $(\mathbb{R}^n, \|\cdot\|_\infty)$. Indeed, show that this isometric embedding is achieved by the map

$$x_j \mapsto (d(x_1, x_j), \dots, d(x_n, x_j)), \qquad 0 \le j \le n.$$