18/Aug/2009

Manjunath Krishnapur

Dvoretsky's theorem: $(B_t)_{0\le t\le 1}$ - Standard 1-dim BM. Then

$$P\left\{ \inf_{0\le t\le 1} \limsup_{h\to 0} \frac{|B_{t+h}-B_t|}{\sqrt{h}} < \frac{1}{10} \right\} = 0$$

Proof: The event in question is the same as $\bigcup_{M=1}^{\infty} A_M$ where

$$A_M = \left\{ \text{for some } t\in[0,1] \text{ and } \forall\, h\in[-1/M, 1/M], \text{ we have } |B_{t+h}-B_t| \le \frac{1}{10}\sqrt{h} \right\}$$

Hence, it suffices to show that $P[A_M]=0$ for each M.

Fix M and let k_0 be large enough so that $2^{-k_0} < \frac{1}{M}$.

Then set $S_{k_0} = \{ I_{k_0,j} \mid 1\le j\le 2^{k_0} \}$ $\left(I_{k_0,j} = \left[\frac{j-1}{2^{k_0}}, \frac{j}{2^{k_0}}\right]\right)$

("initial population")

For $k \ge k_0+1$, we define

$$S_k = \left\{ I_{k,j} \;\middle|\; |B(I_{k,j})| \le \frac{2}{10}\cdot\sqrt{2^{-k}} \text{ and } \operatorname{parent}(I_{k,j}) \in S_{k-1} \right\}$$

$S_{k_0}, S_{k_0+1}, S_{k_0+2}, \ldots$ is a 'branching process'.

Claim 1: The branching process becomes extinct w.p. 1.

That is, $P\left\{ \bigcap_{k=k_0}^{\infty} \{S_k \ne \phi\} \right\} = 0$

Proof of Claim 1: To compute the offspring distributions, condition on $\{B(j/2^{k-1}) \mid 0\le j\le 2^{k-1}\}$ - Values of BM at all dyadics of level $k-1$.

In Levy's Construction, this means we are conditioning on B_{k-1} the k^{th} iterate in the construction.

Recall that $B_k = B_{k-1} + F_k$ (and that $B(t) = B_k(t)$ if t is a dyadic of level k). Thus we can write

$$B\left(\frac{2j+1}{2^k}\right) = \left[\frac{1}{2}\, B\left(\frac{j}{2^{k-1}}\right) + \frac{1}{2}\, B\left(\frac{j+1}{2^{k-1}}\right)\right] + \frac{X_{k,j}}{\sqrt{2^{k+1}}}$$

where $X_{k,j}$, $1 \le j \le 2^{k-1}$, are iid $N(0,1)$ and independent of $\{B(j/2^{k-1}) \mid 0 \le j \le 2^{k-1}\}$. Thus, if $I = \left[\frac{j}{2^{k-1}}, \frac{j+1}{2^{k-1}}\right]$ is a dyadic interval with two 'children' $J = \left[\frac{2j}{2^k}, \frac{2j+1}{2^k}\right]$ and $K = \left[\frac{2j+1}{2^k}, \frac{2j+2}{2^k}\right]$, then,

$$B(J) = \frac{1}{2} B(I) + \frac{X_{k,j}}{\sqrt{2^{k+1}}} \quad ; \quad B(K) = \frac{1}{2} B(I) + \frac{X_{k,j}}{\sqrt{2^{k+1}}}$$

Thus ~~P{B(J)}~~

$$P\left\{ |B(J)| \le \frac{2}{10}\sqrt{2^{-k}} \;\middle|\; B(j/2^{k-1}),\ 0 \le j \le 2^{k-1} \right\}$$

$$= P\left\{ \left|\frac{1}{2} B(I) + \frac{X_{k,j}}{\sqrt{2^{k+1}}}\right| \le \frac{2}{10}\sqrt{2^{-k}} \;\middle|\; B(j/2^{k-1}),\ 0 \le j \le 2^{k-1} \right\}$$

$$\cancel{\le} = P\left\{ \left|\sqrt{2^{k-1}}\, B(I) + X_{k,j}\right| \le \frac{2\sqrt{2}}{10} \;\middle|\; B(j/2^{k-1}),\ 0 \le j \le 2^{k-1} \right\}$$

$$\le \frac{1}{\sqrt{2\pi}} \cdot \frac{4\sqrt{2}}{10}$$

because irrespective of the values of the conditioned values of $\{B(j/2^{k-1}),\ 0 \le j \le 2^{k-1}\}$, we are asking for $X_{k,j}$ (a $N(0,1)$) r.v. to lie in an interval of length $\frac{4\sqrt{2}}{10}$.

(and $X_{k,j}$ is $N(0,1)$ conditional on $\{B(j/2^{k-1}),\ 0 \le j \le 2^{k-1}\}$ because it is independendent of these conditioned variables.)

Note that $S_0, \dots, S_{k-1}$ are functions of $B(j/2^{k-1})$, $0 \le j \le 2^{k-1}$.

Therefore,

$$\cancel{P\{B_J} \quad P\left\{ |B(J)| \le \frac{2}{10}\sqrt{2^k} \,\middle|\, S_0, \dots, S_{k-1} \right\} \le \frac{4\sqrt{2}}{\sqrt{2\pi}\cdot 10} \qquad (\text{Why?})$$

irrespective of the conditioned values of $S_0, \dots, S_{k-1}$.

In particular,

$$P\{ J \in S_k \mid S_0, \dots, S_{k-1} \} = \begin{cases} 0 & \text{if } I \notin S_{k-1}, \\ \le \dfrac{4\sqrt{2}}{\sqrt{2\pi}\cdot 10} & \text{if } I \in S_{k-1}. \end{cases}$$

Summing over all dyadic intervals of level k (need only consider those whose parents are in S_{k-1}) we get

$$\cancel{E[\#S_k] \le \frac{4\sqrt{2}}{\sqrt{\pi}\cdot 10}} \qquad E\left[\#S_k \mid S_0, \dots, S_{k-1} \right] \le \frac{4\sqrt{2}}{\sqrt{2\pi}\cdot 10} \cdot 2 \cdot \#S_{k-1}$$

(Since each $I \in S_{k-1}$ can contribute atmost two offsprings to S_k ....)

Taking Expectations again, we get

$$\begin{aligned} E[\#S_k] &\le \frac{8}{10\sqrt{\pi}} E[\#S_{k-1}] \\ &\ \vdots \ (\text{iterating}) \\ &\le \left(\frac{8}{10\sqrt{\pi}}\right)^{k-k_0} E[\#S_{k_0}] \\ &= 2^{k_0} \cdot \left(\frac{8}{10\sqrt{\pi}}\right)^{k-k_0}. \end{aligned}$$

k_0 is fixed and thus, as $k \to \infty$, $E[\#S_k] \to 0$.

By Markov's inequality, $P\{\#S_k > 0\} \le E[\#S_k] \longrightarrow 0$

$$\Rightarrow \quad P\left\{ \bigcap_{k \ge k_0} \{S_k \ne \phi\} \right\} = 0.$$

This proves Claim 1.

Claim 2: Fix M. $P\{A_M\} = 0$.

Proof of Claim 2: Suppose A_M occurs. Then there is some (random) $t \in [0,1]$ such that ~~B_{t+h}~~

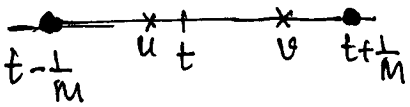
$$\frac{|B_{t+h} - B_t|}{\sqrt{h}} < \frac{1}{10}. \quad \text{Wherever } |h| < \frac{1}{M}.$$

Then, for ~~for~~ any $k \geq k_0$, consider the dyadic interval $I_k(t)$ that contain t. Clearly

$$\left[t - \frac{1}{M}, t + \frac{1}{M}\right] \supseteq I_{k_0}(t) \supseteq I_{k_0+1}(t) \supseteq I_{k_0+2}(t) \dots$$

(the first containment is by the choice of k_0. Rest is obvious)

Further for any interval $[u, v]$ containing t and contained in $[t - 1/M, t + 1/M]$, we have



$$\begin{aligned} |B(v) - B(u)| &\leq |B(v) - B(t)| + |B(t) - B(u)| \\ &\leq \frac{1}{10}\sqrt{v-t} + \frac{1}{10}\sqrt{t-u} \\ &\leq \frac{2}{10} \cdot \sqrt{v-u} \quad \left(\text{Since } \begin{matrix} v-t \leq v-u \\ t-u \leq v-u \end{matrix}\right) \end{aligned}$$

Therefore, for each $k \geq k_0$, we would have

$$|B(I_k(t))| \leq \frac{2}{10}\sqrt{2^{-k}}.$$

This means that ~~the~~ $I_{k_0} \in S_{k_0}$, $I_{k_0+1} \in S_{k_0+1}$, $I_{k_0+2} \in S_{k_0+2} \dots$ implying that the branching process $\{S_k\}$ would survive.

Since by claim-1, survival of $\{S_k\}$ has zero probability, we conclude that $P\{A_M\} = 0$. This proves Claim 2, and hence Dvoretzky's theorem.

Exer: Rework the proof to find the best C_0 for which (this proof shows that)

$$P\left\{ \inf_{t \in [0,1]} \limsup_{h \to 0} \frac{|B_{t+h} - B_t|}{\sqrt{h}} < C_0 \right\} = 0.$$