09/Sep/2009

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Progressive measurability

We showed that the function

$$h : \Omega \times [0,1] \longrightarrow \mathbb{R}$$
$$h(\omega, t) = B_\omega(t)$$

is measurable w.r.t. $\mathcal{F} \otimes \mathcal{B}[0,1]$ on the left and $\mathcal{B}(\mathbb{R})$ on the right.

The same proof shows the following:

Let $(\Omega, \mathcal{F}, P)$ be a prob. space
B - a std. (d-dim) BM
$\mathcal{F}_t = \mathcal{F}_t^0$ or $\mathcal{F}_t^+$ or $\overline{\mathcal{F}}_t^+$

(the statement below is strongest for the smallest filtration, $\mathcal{F}_\cdot^0$)

Then for any $T \geq 0$, the function

$$h_T : \Omega \times [0,T] \longrightarrow \mathbb{R}$$
$$(\mathcal{F}_T \otimes \mathcal{B}[0,T]) \qquad (\mathcal{B}(\mathbb{R}))$$

defined as $h_T(\omega, t) = B_\omega(t)$ is measurable (w.r.t the σ-fields shown in brackets.)

This statement is called progressive measurability of BM.

In class I abandoned the following Claim half way through.

Claim: In the above setting, let $\tau : \Omega \to [0, \infty)$ be an $\mathcal{F}_\cdot$ stopping time. Then $\omega \to B_\omega(\tau(\omega))$ is $\mathcal{F}_\tau$-measurable.

Proof: Consider the functions (for fixed $T \geq 0$)

$$\underset{(\mathcal{F}_T)}{\Omega} \xrightarrow{f} \underset{(\mathcal{F}_T \otimes \mathcal{B}[0,T])}{\Omega \times [0,T]} \xrightarrow{g} \underset{(\mathcal{B}(\mathbb{R}))}{\mathbb{R}}$$

$$f(\omega) = (\omega, \tau(\omega) \wedge T)$$
$$g(\omega, t) = B_\omega(t)$$

g is measurable by progressive measurability of BM.

f is measurable because τ is $\mathcal{F}_\tau$ measurable, whence $\tau \wedge T$ is $\mathcal{F}_T$ measurable. ie,

$$\{\tau \wedge T \leq t\} = \begin{cases} \{\tau \leq t\} & \text{if } t \leq T \\ \{\tau \leq T\} & \text{if } t > T \end{cases}$$

in either case $\{\tau \wedge T \leq t\} \in \mathcal{F}_T$.

The composition, $g \circ f(\omega) = B_\omega(\tau(\omega) \wedge t)$ is therefore $\mathcal{F}_T$ measurable. But

$$\{B_\tau < x\} \cap \{\tau \leq T\} = \underbrace{\{B_{\tau \wedge T} < x\}}_{\in \mathcal{F}_T} \cap \underbrace{\{\tau \leq T\}}_{\in \mathcal{F}_T}$$

$\because g \circ f$ is $\mathcal{F}_T$-measurable. $\qquad \because \tau$ is a stopping time.

Thus $\forall T \geq 0$ $\{B_\tau < x\} \cap \{\tau \leq T\} \in \mathcal{F}_T$

$\Rightarrow B_\tau$ is $\mathcal{F}_\tau$-measurable.