Problem set 1

Due date: 13th Aug

Submit any four

Exercise 1. (1) Let X be a TVS. For any $u \in X$ and $\alpha \in \mathbb{R}$ define $\tau_{u,\alpha}(w) := u + \alpha w$. Show that $\tau_{x,\alpha}$ is an homeomorphism of X with itself. If X is a normed space, and $|\alpha| = 1$, then it is also an isometry.

(2) Let X be a normed linear space. Show that X is a Banach space if and only if the 'unit sphere' $S := \{u \in X : \|u\| = 1\}$ is complete (in the norm-induced metric restricted to S).

Exercise 2. (1) Let X be a Banach space. If $f_n \in X$ and $\sum \|f_n\| < \infty$, then show that $\sum f_n$ converges in X (this means of course, that if we define the partial sums $g_N := \sum_{n \le N} f_n$, then $\|g_N - g\| \to 0$ for some $g \in X$).

(2) Conversely, if X is a normed linear space in which $\sum f_n$ converges in X whenever $\sum \|f_n\| < \infty$, then show that X is a Banach space.

Exercise 3. Let $(M, \mathcal{F}, \mu)$ be a finite measure space. Assume that $f \in L^\infty(\mu)$ (then $f \in L^p(\mu)$ for all p). If $t_n = \int |f|^n d\mu$, then show that $\frac{t_{n+1}}{t_n} \to \|f\|_\infty$. Hence or otherwise, show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. [**Note:** This justifies calling the esential supremum as the L^∞ norm.]

Exercise 4. Let $\mathbb{D}$ be the open unit disk in the complex plane. Let $H^\infty(\mathbb{D})$ denote the set of all *bounded* holomorphic functions on $\mathbb{D}$ with the nom $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$. Show that $H^\infty(\mathbb{D})$ is a Banach space.

Exercise 5. (1) Let V be a symmetric ($\mathbf{x} \in V$ implies $-\mathbf{x} \in V$) open neighbourhood of the origin in $\mathbb{R}^n$. Define $\|\mathbf{x}\|_V = \inf\{r > 0 : r^{-1}\mathbf{x} \in V\}$. Show that $\|\cdot\|_V$ is a norm on $\mathbb{R}^n$ if and only if V is convex and bounded. Here "bounded" is in the sense of the standard Euclidean metric.

(2) Show that $\|\mathbf{x}\| := (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$ is *not* a norm for $0 < p < 1$.

[**Remark:** In a normed space, if $V = \{u : \|u\| < 1\}$, then clearly $\|\cdot\| = \|\cdot\|_V$. The idea is whether we can make any TVS into a normed space by fixing an arbitrary V as the unit ball, and then defining the norm as $\|\cdot\|_V$? The problem shows that even in $\mathbb{R}^n$ this does not always work.]

Exercise 6. For $1 \le p \le \infty$, let L^p denote the Lebsgue space $L^p([0,1], \mathcal{B}, m)$ where $\mathcal{B}$ is the Borel sigma-algebra and m is the Lebesgue measure. Show the completeness of L^p for $p < \infty$ by completing the following steps.

(1) If $g_n \in L^1$ and $\sum \int |g_n|\, dm < \infty$, show that $\sum g_n$ converges a.s.

(2) Given a sequence f_n that is Cauchy in L^p, show that there is subsequence $\{n_k\}$ such that f_{n_k} converges a.s. to some f. [**Hint:** One can write $f_{n_k} = f_{n_1} + (f_{n_2} - f_{n_1}) + \ldots + (f_{n_k} - f_{n_{k-1}})$. This suggest choosing n_k so that part (1) can be applied].

(3) Argue that $f \in L^p$ and that $f_n \xrightarrow{L^p} f$.