Problem set 2
Due date: 5th Sep

Submit any four

Exercise 7. Let $\mathcal{S}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Recall that the norm on the quotient space $Q = X\Big|_{\mathcal{S}}$ is defined as $\|[u]\|_Q := \inf_{v\in[u]} \|v\|$. Verify that the parallellogram law holds for $\|\cdot\|_Q$ and hence Q is a Hilbert space. [**Extra:** How to express the inner product on Q in terms of the inner product on $\mathcal{H}$?]

Exercise 8. [*Riesz representation theorem - alternate proof*] Let $\mathcal{H}$ be a Hilbert space. If L is a bounded linear functional on $\mathcal{H}$, show that there exists $v \in \mathcal{H}$ such that $Lu = \langle u, v\rangle$ for all $u \in \mathcal{H}$ as follows.

(1) Show that there exists a unit vector $v \in \mathcal{H}$ such that $Lv = \|L\|$. [**Hint:** Get unit vectors v_n such that $Lv_n \to \|L\|$. Show that v_n is a Cauchy sequence by applying the parallellogram law to v_n and v_m.]

(2) Let $w = \|L\|v$. Show that $Lu = \langle u, w\rangle$ for all $u \in H$.

Exercise 9. In $L^2[0,1]$, consider the vectors[1] $f_{n,k} = \mathbf{1}_{[k2^{-n},(k+1)2^{-n}]}$ for $n \geq 0$ and $0 \leq k \leq 2^n - 1$. Apply Gram-Schmidt process to the functions $f_{0,0}, f_{1,0}, f_{1,1}, f_{2,0}, f_{2,1}, f_{2,2}\ldots$ to get an orthonormal set in $L^2[0,1]$. Give the end result explicitly and show that this orthonormal set is an orthonormal basis.

Exercise 10. Let μ be a compactly supported Borel measure on $\mathbb{R}$ so that polynomials are dense in $L^2(\mu)$. . Then, $m_n(x) := x^n \in L^2(\mu)$.

(1) Show that $\{m_0, m_1, m_2, \ldots\}$ is a linearly dependent set if and only if μ is supported on finitely many points.

(2) Assume that the support of μ is not finite. Then apply Gram-Schmidt to $m_0, m_1, m_2, \ldots$ to get polynomials $p_0, p_1, p_2\ldots$ such that $\deg(p_k) = k$ and $\int p_k p_\ell = \delta_{k,\ell}$. Show that $\{p_k : k \geq 0\}$ is an ONB for $L^2(\mu)$. These are called orthogonal polynomials with respect to μ.

(3) Show that there are real numbers a_n, b_n such that the *three term recurrence relationship* $xp_n(x) = b_{n-1}p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x)$ for all $n \geq 0$. Here we take $p_{-1} \equiv 0$ and $b_{-1} = 0$. [**Hint:** Consider the expansion of $xp_n(x)$ in the ONB $\{p_k\}$].

Exercise 11. Let $u_1, \ldots, u_n$ be vectors in a Hilbert space $\mathcal{H}$. Define the $n \times n$ *Gram matrix* $A := \big(\langle u_i, u_j\rangle\big)_{i,j\leq n}$ whose (i, j) entry is $\langle u_i, u_j\rangle$.

(1) Show that A is non-negative definite. Show that A is singular if and only if $u_1, \ldots, u_n$ are linearly dependent.

(2) For $k \leq n$ let $\mathcal{M}_k = \text{span}\{u_1, \ldots, u_k\}$. Then, show that

$$|\det(A)| = \|u_1\|\|P^{\perp}_{\mathcal{M}_1} u_2\|\|P^{\perp}_{\mathcal{M}_2} u_3\|\ldots\|P^{\perp}_{\mathcal{M}_{n-1}} u_n\|.$$

(3) Prove *Hadamard's inequality* $|\det(A)| \leq \prod_{k=1}^{n} \|u_k\|$.

[1] Always $\mathbf{1}_A$ denotes the inicator function of the set A, that is $I_A(x) = 1$ if $x \in A$ and 0 otherwise.