Problem set 4

Due date: 5th Oct

Exercise 15. Let X,Y be normed linear spaces. Recall that the direct sum $X \oplus Y := \{(u,v) \,:\, u \in X,\ v \in Y\}$ with $\|(u,v)\| := \sqrt{\|u\|_X^2 + \|v\|_Y^2}$ is also a normed linear space.

(1) Show that $(X \oplus Y)^* \cong X^* \oplus Y^*$ (isometric as Banach spaces).

(2) Use part 1 to give example of a Banach space X such that X^* is isometrically isomorphic to X, but X is not a Hilbert space.

Exercise 16. (1) Let A be a subset of a normed linear space X. If each bounded linear functional is bounded on A (that is, $\sup_{u \in A} |Lu| < \infty$ for each $L \in X^*$), then show that A is bounded in X (that is, $\sup_{u \in A} \|u\| < \infty$).

(2) (**Another version of UBP:**) Let X be a Banach space and Y a normed linear space. Let T_i, $i \in I$ be a collection of bounded linear transformations from X to Y. Suppose $\sup_{i \in I} |L(T_i(u))| < \infty$ for each $u \in X$ and each $L \in Y^*$, then show that $\{T_i \,:\, i \in I\}$ is uniformly bounded, that is $\sup_{i \in I} \|T_i\| < \infty$.

Exercise 17. (1) Fix a sequence $\alpha_1, \alpha_2, \ldots$ be an unbounded sequence. Let $A_\alpha = \{x \in \ell^1 \,:\, \sum \alpha_n x_n \text{ converges}\}$. Show that A_α is of first category (under the ℓ^1 metric).

(2) Show that ℓ^2 is of first category in ℓ^1 (here we are thinking of ℓ^2 as a subset of ℓ^1, and the metric we use is the ℓ^1 metric).

Exercise 18. (**Hormander**). Show that there is a finite constant C such that $\|f'\| \le C(\|f\| + \|f''\|)$ for any $f \in C^2$. Here $\|\cdot\|$ denotes the supremum norm. [**Remark:** We know that it is not possible to bound $\|f'\|$ in terms of $\|f\|$ alone, or in terms of $\|f''\|$ alone, but using both, we can. The hint is to consider f' as a function of the pair (f, f'').]

Exercise 19. Let X be a Banach space and let M,N be closed subspaces of X. Suppose that M,N are complementary in the sense that every $u \in X$ can be written in a unique way as $u_1 + u_2$ with $u_1 \in M, u_2 \in N$. Thus we may define the "projections" $P : X \to M$ and $Q : X \to N$ by $P(u) = u_1$ and $Q(u) = u_2$. Show that P, Q are bounded linear operators.

Exercise 20. [**Moment problem on** S^1] Given complex numbers α_n, $n \in \mathbb{Z}$, we want to find necessary and sufficient conditions on the $(\alpha_n)_n$ under which there exists a measure μ on $S^1 = [0, 2\pi)$ such that $\int e^{int} d\mu(t) = \alpha_n$ for all $n \in \mathbb{Z}$. Recall that a necessary condition is that

$$(\star) \qquad \sum_{k,\ell} z_k \overline{z_\ell} \alpha_{k+\ell} \ge 0, \qquad \text{for any } z_k s \text{ where all but finitely many } z_k \text{ are zeros.}$$

Prove that this condition is sufficient by following these steps.

(1) (**Fejér and F.Riesz**). If $p(t) = \sum_{k=-n}^{n} c_k e^{ikt}$ is a real trigonometric polynomial (so $c_{-k} = \overline{c_k}$ and we may assume $c_n \neq 0$) such that $p(t) \ge 0$ for all $t \in [0, 2\pi)$, then show that there exists a trigonometric polynomial $q(t)$ such that $p(t) = |q(t)|^2$. [**Remark:** For the homework, you may omit this part, and just assume it for the rest of the exercise. If you want to prove it, consider the related polynomial $P(z) = c_n + c_{-n+1} z + \ldots + c_n z^n$, show that $z^{2n} \overline{P(1/\overline{z})} = P(z)$. Further, assume that p is strictly positive on S^1 and analyze the roots of P - are there any on the circle, how are those off the unite circle related? In the end argue how you would eliminate strict positivity condition.]

(2) Consider the subspace $M = \text{span}\{e^{int} \,:\, n \in \mathbb{Z}\}$ of $C(S^1)$ and imitate the solution of Hausdorff moment problem given in class to find a measure μ.