Problem set 5
Due date: 19th Oct

Exercise 21. Let X be a normed linear space and let X^* be its dual.

(1) Show that X with its weak topology is a topological vector space. Show that X^* with its weak* topology is also a topological vector space.

(2) Let X_w denote X with the weak topology. Show that X_w^* is equal to X^*. Here, since X_w is not normed, the dual X_w^* should be defined as the space of continuous linear functionals, not bounded linear functionals.

(3) Let $X_{w^*}^*$ denote X^* with the weak* topology. Show that its dual is X.

Exercise 22. For each of the following statements, determine whether they are true or false and give a proof or counterexample accordingly. Everywhere X is a Banach space and weak and weak* refer to the topologies on X and X^* as usual.

(1) We have seen that if $u_n \xrightarrow{w} u$, then $\{u_n\}$ is norm-bounded. The same is true for nets, that is, if $u_\alpha \xrightarrow{w} u$, then $\{\|u_\alpha\|\}$ is bounded.

(2) $C[0,1]$ is dense in L^∞ in (a) norm topology on L^∞. (b) weak* topology (induced on L^∞ as the dual space of L^1).

(3) The unit ball in X is compact in weak topology induced by X^*.

(4) The closure of $\{u \in X \,:\, \|u\| = 1\}$ in weak topology is $\{u \in X \,:\, \|u\| \leq 1\}$. The closure of $\{L \in X^* \,:\, \|L\| = 1\}$ in weak* topology is $\{L \in X^* \,:\, \|L\| \leq 1\}$.

(5) Let X, Y be Banach spaces and let X_w, Y_w denote the same spaces with their weak topologies. Let $T : X \to Y$ be a bounded linear operator. Then, the following operators are continuous: (a) $T : X \to Y_w$. (b) $T : X_w \to Y$. (c) $T : X_w \to Y_w$.

Exercise 23. (1) Let X be a normed linear space. Say that $A \subseteq X$ is weakly bounded if $\{Lu \,:\, u \in A\}$ is bounded for each $L \in X^*$. Show that A is weakly bounded if and only if it is norm-bounded.

(2) Let H be a Hilbert space and suppose $u_n \xrightarrow{w} u$ where $u_n, u \in H$. Then, show that $u_n \xrightarrow{\|\cdot\|} u$ if and only if $\|u_n\| \to \|u\|$.

Exercise 24. (<u>*Extra: Need not submit*</u>). Banach-Alaoglu theorem gives certain compact subsets in weak* topologies (and hence also in weak topologies in some cases). Here you determine compact subsets in norm-topology in certain specific spaces.

(1) If $A \subseteq C[0,1]$, Arzela-Ascoli theorem says that A is precompact if and only if A is uniformly bounded and equicontinuous.

(2) In ℓ^2 the *Hilbert cube* $A = \{\mathbf{x} \,:\, 0 \leq x_k \leq \frac{1}{k}\}$ is compact.

(3) If $A \subseteq \ell^p$, $1 \leq p < \infty$, show that A is precompact if and only if A is uniformly bounded (in L^2 norm) and has uniformly decaying tails (this means that given $\varepsilon > 0$, there exists $N < \infty$ such that for every $\mathbf{x} \in A$ we have $\sum_{i \geq N} |x_i|^p < \varepsilon$.