PROBLEMS IN MEASURE THEORY

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Note: These are largely problems I put out when teaching the course on measure and integration at IISc during Jan-Apr 2018. While a few are made up, most are taken them from various books, for example, from Bogachev's and Dudley's well-known books on the subject. Also I borrowed freely from Kavi Ramamurthy's problem sets (`http://www.isibang.ac.in/~adean/infsys/database/index.htm`)

Problem 1. In each of the following cases, find $\sigma(S)$ and $\mathcal{A}(S)$ (the sigma-algebra and algebra generated by S).

(1) X is a set and S is the collection of all singleton subsets of X.

(2) X is a set and S is the collection of all two-element subsets of X.

(3) $A_1, A_2, \ldots$ are pairwise disjoint sets of X such that $\bigcup\limits_n A_n = X$.

(4) Do the previous exercise if A_i are pairwise disjoint but their union may not equal X.

Problem 2. Let $\mathcal{F}$ be a sigma algebra on X. Let $A_1, A_2, \ldots$ be elements of $\mathcal{F}$. Show that the following sets are also in $\mathcal{F}$ (first express the set in proper notation).

(1) The set of $x \in X$ that belong to exactly five of the A_ns.

(2) The set of $x \in X$ that belong to all except five of the A_ns.

(3) The set of $x \in X$ that belong to infinitely many of the A_ns.

(4) The set of $x \in X$ that belong to all but finitely many of the A_ns.

Problem 3. Decide whether the following statements are true or false and justify your answer.

(1) A finite union of σ-algebras is not necessarily a σ-algebra.

(2) If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ is an increasing sequence of sigma-algebras of a set X, then $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is also a sigma-algebra.

(3) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : X \mapsto Y$ be a function. Then $\mathcal{G} := \{T(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y.

(4) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : Y \mapsto X$ be a function. Then $\mathcal{G} := \{T^{-1}(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y. Here $T^{-1}(A) = \{y \in Y : T(y) \in A\}$.

(5) There is no sigma-algebra with exactly 1000 elements.

Problem 4. Let $\mathcal{F} = \sigma(S)$ where S is a collection of subsets of X. Suppose $a, b \in X$ are such that every set in S either contains both a and b or does not contain either. Then show that the same property holds for sets in $\mathcal{F}$.

Problem 5. Consider the outer Lebesgue measure λ_2^* on $\mathbb{R}^2$. Show that $[a, b] \times [c, d]$ in Lebesgue measurable by checking the Carathéodary cut condition.

Problem 6. Suppose $A \subseteq B$ are (Lebesgue) measurable subsets of R and $\lambda(A) = \lambda(B)$. Then show that any set C such that $A \subseteq C \subseteq B$ is also measurable and that $\lambda(C) = \lambda(A)$.

Problem 7. (*) Let (X, d) be a compact metric space. Fix $\alpha > 0$ and define

$$\mu_\alpha^*(A) := \liminf_{\delta \downarrow 0} \left\{ \sum_{k=1}^{\infty} \text{dia}(B_j)^\alpha : B_j \text{ are open balls in } X \text{ such that } \text{dia}(B_j) < \delta \text{ and } \bigcup_j B_j \supseteq A \right\}.$$

Show that μ_α^* is an outer measure.

When $X = \mathbb{R}$ with the usual metric, μ_α^* is the same as λ^* (the Lebesgue outer measure) when $\alpha = 1$. Further, μ_α^* is zero if $\alpha > 1$ and $\mu_\alpha^*(A) = \infty$ if $\alpha < 1$ and A is any interval.

Problem 8. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(1) If $A_n, A \in \mathcal{F}$ and $A_n \uparrow A$, show that $\mu(A_n) \uparrow \mu(A)$.

(2) If $A_n, A \in \mathcal{F}$ and $A_n \downarrow A$ and $\mu(A_n) < \infty$ for some n, then show that $\mu(A_n) \downarrow \mu(A)$.

(3) Show that the second conclusion may fail if $\mu(A_n) = \infty$ for all n.

Problem 9. A measure μ on $(X, \mathcal{F})$ is said to be σ-finite if there exist $E_1, E_2, \ldots$ in $\mathcal{F}$ such that $X = \bigcup_n E_n$ and $\mu(E_n) < \infty$ for all n.

(1) Show that a σ-finite measure space has sets of arbitrarily high but finite measure.

(2) Show that a σ-finite measure has at most countably many atoms. Show that the previous assertion is false without the σ-finiteness assumption.

Problem 10. Let $\mathcal{F} = \sigma(S)$ be a sigma algebra on X. Show that for any $A \in \mathcal{F}$, there exists countably many sets $A_1, A_2, \ldots$ in S such that $A \in \sigma(\{A_1, A_2, \ldots\})$.

Problem 11. Let $\mathcal{F}$ be a sigma algebra on X and assume that $B \subseteq X$ is not in $\mathcal{F}$. Show that the smallest sigma algebra containing $\mathcal{F}$ and B is the collection of all sets of the form $(A_1 \cap B) \cup (A_2 \cap B^c)$ where $A_1, A_2 \in \mathcal{F}$.

(*) If μ is a measure on $\mathcal{F}$, can you extend it to $\mathcal{G}$ in some way? Is the extension unique?

Problem 12. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $k.\mathbb{N} = \{k, 2k, 3k, \ldots\}$. What is the sigma algebra generated by (1) the collection $k.\mathbb{N}$, $k \geq 1$? (2) the collection $p.\mathbb{N}$ as p runs over all primes?

Problem 13. (*) Show that any convex set in $\mathbb{R}^d$ is (Lebesgue) measurable. Is it necessarily Borel measurable?

Problem 14. If $A \subseteq \mathbb{R}$, then there exists a Lebesgue measurable set B that contains A and such that $\lambda(B) = \lambda^*(A)$. Can we choose B to be Borel measurable?

Problem 15. (*) If $A \subseteq \mathbb{R}$ is measurable and has positive Lebesgue measure, show that it contains a three-term arithmetic progression, i.e., there exist $a, b \in A$ such that $\frac{1}{2}(a + b) \in A$.

Problem 16. Let E be a measurable subset of $\mathbb{R}$. Let $L(E)$ be the supremum of all t such that there is a surjective Lip(1) function $f : E \mapsto [0, t]$. Show that $L(E) = \lambda(E)$.

Problem 17. Let $\mathcal{A}' \subseteq \mathcal{A}$ be two algebras on X that genberate the same sigma algebra (call it $\mathcal{F}$). Now suppose we have a countably additive measure μ on $\mathcal{A}$ and let μ' be the restriction of μ to $\mathcal{A}'$.

By the theorem proved in class, both μ' and mu extend to $\mathcal{F}$ as measures. Are the Carathéodary sigma algebras the same? If yes, are the extended measures equal? On the way, is the outer measure constructed from μ and μ' the same?

As a particular case, if we start with a measure μ on a sigma algebra $\mathcal{F}$, and extend it (since $\mathcal{F}$ is also an algebra), what is the extended sigma algebra? Is it $\mathcal{F}$ or is it larger?

Problem 18. Let μ be a Borel measure on $\mathbb{R}$ such that $\mu\{x\} = 0$ for each $x \in \mathbb{R}$. Show that for any $\epsilon > 0$, there exists a dense open set $U_\epsilon \subseteq \mathbb{R}$ such that $\mu(U_\epsilon) < \epsilon$.

Problem 19. Let $X = \{0, 1\}^{\mathbb{N}}$ be the sequence space of zeros and ones. An element $\omega \in X$ is written as $\omega = (\omega_1, \omega_2, \ldots)$.

(1) A cylinder set is one defined by specifying the values of finitely many co-ordinates. Eg., $\{\omega : \omega_1 = 0, \omega_2 = 1, \omega_7 = 1\}$. Show that the complement of a cylinder set is a finite union of pairwise disjoint cylinder sets. Use this to describe $\mathcal{A}(S)$ as the collection of all finite unions of pairwise disjoint cylinder sets.

(2) If A is a cylinder set for which exactly n co-ordinate values are specified, define $\mu_0(A) = 2^{-n}$. Extend in the obvious way to $\mathcal{A}(S)$ and show that mu_0 is countably additive on $\mathcal{A}(S)$.

(3) Argue that there is a measure μ on $\sigma(S)$ that extends μ_0.

[*Note:* This exercise is to make precise the notion of a infinite sequence of fair coin tosses.]

Problem 20. Let A and B be measurable subsets of the line.

(1) Show that $\lambda(A+B) \geq \lambda(A)+\lambda(B)$. [*Note:* The discrete version of this is: If A and B are subsets of $\mathbb{Z}$, then $|A+B| \geq |A|+|B|-1$, where $|A|$ denotes the cardinality of A.]

(2) Show that there can be no reverse inequality by constructing A, B having zero measure but such that $A+B=\mathbb{R}$.

Problem 21. Let A, B be measurable subsets of the real line.

(1) If A and B have positive measure, show that $A-B$ contains an interval.

(2) If $x_n + A = A$ for a sequence $x_n \to 0$, then show that $\lambda(A) = 0$ or $\lambda(A^c)=0$.

Problem 22. Let A be a bounded subset of $\mathbb{R}$ with positive Lebesgue measure. Given are $\epsilon > 0$ and a sequence δ_n converging to zero. Show that there exists $B \subseteq A$ and a subseequnce δ_{n_k} such that (a) B is measurable and $\lambda(B) > \lambda(A) - \epsilon$, (b) if $x \in B$ then $x \pm \delta_{n_k} \in B$ for all k.

Problem 23. Let A be a measurable subset of $\mathbb{R}$. If $\lambda_1(A) > 0$, show that $(A+\mathbb{Q})^c$ has zero Lebesgue measure.

Problem 24. If K is a compact subset of $\mathbb{R}^d$, the the set $\{x \in \mathbb{R}^d : d(x,K)=1\}$ has zero Lebesgue measure.

Problem 25. Let A be a measurable subset of $\mathbb{R}^2$ with positive Lebesgue measure. Show that there is a measurable set B having positive measure and a number $\delta > 0$ such that $B, B+(\delta,0), B+(0,\delta), B+(\delta,\delta)$ are all subsets of A.

Problem 26. If $A \subseteq \mathbb{R}^2$ is measurable, then $\lambda_2(A) = \inf \sum_n |B_n|$ where the infimum is over all countable coverings of A by open balls $B_1, B_2, \ldots$ and $|B_n|$ is the area. Deduce that Lebesgue measure on $\mathbb{R}^2$ is rotation-invariant.

[*Note:* We defined Lebesgue measure using coverings by open rectangles (with sides parallel to the axes). This exercise is to show that the shape of the basic sets does not affect the measure we get].

Problem 27. True or false?

(1) If $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is continuous and $A \subseteq \mathbb{R}^2$ is measurable, then $f(A) := \{f(x) : x \in A\}$ is also measurable.

(2) Let $\mu : \mathcal{A} \mapsto [0,\infty]$ be a finitely additive set function on an algebra $\mathcal{A}$. Then countable additivity of μ is equivalent to continuity of μ under monotone limits (i.e., $A_n \uparrow A$, $A_, A \in \mathcal{A}$ implies $\mu(A_n) \uparrow \mu(A)$ and similarly for decreasing limits).

(3) $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on X. If μ is a σ-finite measure on $\mathcal{F}$ then its restriction to $\mathcal{G}$ is also σ-finite.

Problem 28. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. Show that any of the following conditions implies that f is Borel measurable.

(1) f is increasing.

(2) f is right-continuous.

(3) f is lower semi-continuous (means that $f(x) = \liminf\limits_{y\to x} f(y)$ for all x).

Problem 29. Let $(X, \mathcal{F})$ be a measure space and let $f_n : X \mapsto \mathbb{R}$ be a sequence of Borel measurable functions.

(1) Show that $\sup_n f_n$, $\limsup f_n$, $\lim f_n$ (assuming limit exists), $\sum_n f_n$ (assuming that the sum converges pointwise), are measurable (where necessary, these functions may be allowed to take values $\pm\infty$ also).

(2) Show that the set $\{x \in X : \lim_n f_n(x) \text{ exists }\}$ is measurable in X. Same for $\{x : \limsup_n f_n(x) = +\infty\}$, $\{x : \lim f_n(x) = +\infty\}$.

(3) Show that the supremum of an uncountable family of measurable functions need not be measurable (even if the supremum is finite everywhere).

Problem 30. Let (X, Σ) be a measure space and suppose $f_1, f_2 : X \mapsto \mathbb{R}$. Then $F = (f_1, f_2) : X \mapsto \mathbb{R}^2$.

(1) Show that F is Borel measurable if and only if both f_1, f_2 are Borel measurable.

(2) Deduce that if f_1, f_2 are measurable, then so are $af_1 + bf_2$ (for any $a, b \in \mathbb{R}$) and $f_1 f_2$ and f_1/f_2 (for the last one, assume that $f_2 \geq 0$ so that f_1/f_2 can be defined unambiguously as an $\overline{\mathbb{R}}$-valued function).

(3) If $f : \mathbb{R} \mapsto \mathbb{R}$ is pointwise differentiable, then show that $f' : \mathbb{R} \mapsto \mathbb{R}$ is measurable.

Problem 31. Find a measurable function on an appropriate interval in $\mathbb{R}$ so that the push-forward of Lebesgue measure of the interval is the measure μ satisfying

(1) $\mu(a,b] = \int_a^b e^{-|x|}dx$ for $a < b$.

(2) $\mu(a,b] = \int_a^b \frac{1}{1+x^2}dx$ for $a < b$.

(3) $\mu(a,b] =$ number of integers in $(a,b]$.

Problem 32. Let K be the $1/3$-Cantor set.

(1) Write out in detail the sketch given in class that there is a bijection $T : [0,1] \mapsto K$ such that T and T^{-1} are Borel measurable.

(2) Use T to construct a Lebesgue measurable set in $\mathbb{R}$ that is not Borel measurable. [*Hint:* Use the existence of non-measurable sets.]

Problem 33. (*) This exercise is to show the isomorphism between $((0,1), \mathcal{B}_{(0,1)}, \lambda_1)$ and $((0,1)^2, \mathcal{B}_{(0,1)^2}, \lambda_2)$. As pointed out in class, the essential idea is to take a number $x = 0.x_1x_2...$ in binary expansion and map it to (y,z) where $y = 0.x_1x_3...$ and $z = 0.x_2x_4....$ But because of the ambiguities of binary expansion at dyadic rationals, this is not quite a bijection between $(0,1)$ and $(0,1)^2$. Here is how to fix this.

(1) Find $T_1 : (0,1) \mapsto (0,1)^2$ that is injective and such that $\mathrm{Im}(T_1)$ is a Borel set in $(0,1)^2$.

(2) Find $T_2 : (0,1)^2 \mapsto (0,1)$ that is injective and such that $\mathrm{Im}(T_2)$ is a Borel set in $(0,1)$.

(3) Use the idea of the proof of Schroder-Bernstein theorem to get a bijection $T : (0,1) \mapsto (0,1)^2$ so that T, T^{-1} are Borel measurable.

(4) If you base your T_2 on the binary expansion idea above, it turns out that Lebesgue measure on the two spaces are pushed forward to each other by T and T^{-1}.

Problem 34. (**16 marks**) State True or False and justify accordingly.

(1) If S is a collection of subsets of X, then $\mathcal{A}(S) = \bigcup_F \mathcal{A}(F)$ where the union is over all *finite* subsets $F \subseteq S$.

(2) If A is a bounded, measurable subset of $\mathbb{R}$, then $\lambda(A) = \inf\{\lambda(C) : C \supseteq A,\ C \text{ is closed}\}$.

(3) For $A \subseteq \mathbb{R}^2$, let $\Pi(A) = \{x : (x,y) \in A \text{ for some } y\}$ (a subset of $\mathbb{R}$). If A is measurable, then so is $\Pi(A)$.

(4) If A is a measurable subset of $\mathbb{R}^d$ and $0 \leq x \leq \lambda_d(A)$, then there is a measurable subset $B \subseteq A$ such that $\lambda_d(B) = x$.

Problem 35. (**10 marks**) Let $A = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, \text{ and } x + y < 1\}$. Just from the definition of the outer measure, calculate $\lambda_2(A)$.

Problem 36. (**10 marks**) Let S be the collection of all intervals of the form $(-a,a)$ with $a \in \mathbb{R}$. Show that if $A \in \sigma(S)$ and $x \in A$, then $-x \in A$ (in short, $A = -A$).

Problem 37. (**10 marks**) Let X and Y be metric spaces and let $f : X \mapsto Y$ be a continuous function. If A is a Borel set in Y, show that $f^{-1}(A)$ is a Borel set in X.

Problem 38. (**10 marks**) Let A and B be measurable subsets of the line with positive measure. Show that there exists some $x \in \mathbb{R}$ such that $A \cap (B + x)$ is an infinite set.

Problem 39. Suppose the set of discontinuity points of $f : \mathbb{R} \mapsto \mathbb{R}$ has has zero Lebesgue measure. Show that f is a measurable function (w.r.t the Lebesgue measure).

Problem 40. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is such that at each point, f is either right continuous or left continuous (or both). Is f necessarily Borel measurable?

Problem 41. Let $(X, \mathcal{F}, \mu)$ be a measure space. If f $f : X \mapsto \mathbb{R}$ is a bounded measurable function. Show that there exist simple functions s_n such that $s_n \to f$ uniformly on X.

Problem 42. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{S}_+$.

(1) Show that $t \mapsto \mu\{f > t\}$ is measurable from $[0, \infty]$ to $[0, \infty]$.

(2) Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\} dt$.

(3) What about $\int_0^\infty \mu\{f \geq t\} dt$?

Problem 43. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Define $\nu : \mathcal{F} \mapsto [0, \infty]$ be defined by $\nu(A) = \int_X f \mathbf{1}_A d\mu$. Show that ν is a measure.

Problem 44. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Show that for any $t > 0$ we have $\mu\{f \geq t\} \leq \frac{1}{t} \int_X f d\mu$ (Markov's inequality).

Problem 45. Consider the space of Riemann integrable functions on $[0, 1]$ with (pseudo)-metric $d(f, g) = \int_0^1 |f - g|$. Show that this metric space is not complete.

Problem 46. Suppose f, g are integrable functions on a measure space $X, \mathcal{F}, \mu)$. Which of the following are necessarily integrable? (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, (e) $f \vee g$, (f) $f \wedge g$.

Problem 47. Given an integrable function f on $(X, \mathcal{F}, \mu)$, show that there exist simple functions s_n such that $\int_X |f - s_n| d\mu \to 0$ as $n \to \infty$.

Problem 48. (1) If f_n are non-negative measurable functions on $(X, \mathcal{F}, \mu)$ and $f = \sum_n f_n$, then show that $\int_X f d\mu = \sum_n \int_n f_n d\mu$.

(2) If g is a non-negative measurable function and $\nu(A) := \int_A g d\mu$ for $A \in \mathcal{F}$, (convention: By $\int_A f d\mu$ we just mean $\int_X f \mathbf{1}_A d\mu$), then show that ν is a measure on $\mathcal{F}$.

Problem 49. Suppose $f : (a, b) \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that (a) $\theta \mapsto f(\theta, x)$ is differentiable for each $x \in \mathbb{R}$, (b) $x \mapsto f(\theta, x)$ is integrable for each $\theta \in (a, b)$, (c) $|f(\theta_1, x) - f(\theta_2, x)| \leq g(x)|\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in (a, b)$ and $x \in \mathbb{R}$ and that g is integrable over $\mathbb{R}$ (w.r.t. Lebesgue measure). Define $H(\theta) := \int_{\mathbb{R}} f(\theta, x) d\lambda(x)$.

Show that H is differentiable, that $\frac{d}{d\theta} f(\theta, x)$ is integrable over $\mathbb{R}$ for each θ, and that

$$\frac{d}{d\theta} H(\theta) = \int_{\mathbb{R}} \frac{d}{d\theta} f(\theta, x) d\lambda(x).$$

Problem 50. Assume that f_n are non-negative measurable functions that are bounded above by an integrable function g a.e.[μ] (i.e. $0 \leq f_n \leq g$ a.e.[μ]).

(1) Show that $\limsup_{n\to\infty} \int_X f_n d\mu \leq \int_X (\limsup_{n\to\infty} f_n) d\mu$.

(2) If $f_n \downarrow f$ a.e.[μ], and $\int_X f_n d\mu$ is finite for some n, then show that $\int_X f_n d\mu \downarrow \int_X f d\mu$.

Problem 51. If f is an integrable function, show that $\int |f| \mathbf{1}_{|f|>n} d\mu \to 0$. More generally, if A_n is any sequence of events such that $\mu(A_n) \to 0$, then show that $\int_{A_n} |f| d\mu \to 0$ as $n \to \infty$.

Problem 52. If $f : [a, b] \mapsto \mathbb{R}$ is a continuous function, show that its Riemann integral is equal to its Lebesgue integral (w.r.t. Lebesgue measure on $[a, b]$). [*Note:* This is true more generally.]

Problem 53. Let A_n be measurable sets in $(X, \mathcal{F}, \mu)$. Let A be the set of all x that belong to A_n for infinitely many n (is A measurable?).

(1) If $\sum_n \mu(A_n) < \infty$, then show that $\mu(A) = 0$.

(2) Show that $\mu(A_n) \to 0$ does not necessarily imply that $\mu(A) = 0$.

Problem 54. If f_n are non-negative measurable functions with $\int_X f_n d\mu = 1$ for all n and $f_n \to f$ a.e.[μ], then show that the set of all possible values of $\int_X f d\mu$ is $[0, 1]$.

Problem 55. Suppose f_n, f are non-negative integrable functions on $(X, \mathcal{F}, \mu)$ and that $f_n \to f$ a.s.[μ]. Show that $\int_X f_n d\mu - \|f_n - f\|_{L^1(\mu)} \to \int_X f d\mu$.

Problem 56. State whether true or false and justify. Whenever we write L^p, it is assumed that $1 \leq p \leq \infty$.

(1) $L^\infty(\mathbb{R}, \mathcal{B}, \lambda_1)$ is not separable (has no countable dense subset).

(2) If $p_1 < p_2$, then $L^{p_1}(X, \mathcal{F}, \mu) \supseteq L^{p_2}(X, \mathcal{F}, \mu)$.

(3) If $p_1 < p_2 < p_3$ then $L^{p_1}(\mu) \cap L^{p_3}(\mu) \subseteq L^{p_2}(\mu)$.

(4) If μ is a finite measure, then for any measurable f, the set $\{p \geq 1 : f \in L^p(\mu)\}$ is an interval of the form $[1, r]$ or $[1, r)$ for some $r \leq \infty$.

(5) If f is a measurable function on $(X, \mathcal{F}, \mu)$, then the set $\{p \geq 1 : f \in L^p(\mu)\}$ is an interval (Extra: Give exmples to show that this interval can be open or closed on the left or right).

(6) $L^1(\mu) \cap L^2(\mu)$ is complete in the norm $\| \cdot \|_1 + \| \cdot \|_2$.

Problem 57. Show that $L^\infty(X, \mathcal{F}, \mu)$ is complete (This was skipped in class).

Problem 58. Let $f \in L^p(X, \mathcal{F}, \mu)$ with $p < \infty$.

(1) Show that $\|f\|_p^p = \int_0^\infty pt^{p-1}\mu\{|f| > t\}dt$. [*Note:* This was shown for $p = 1$ and simple non-negative f in an earlier exercise.]

(2) Show that that if $f \in L^p(\mu)$, then $\mu\{|f| > t\} \leq \|f\|_p^p t^{-p}$.

Problem 59. Let f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$.

(1) Show that $\mu\{f > 0\} \geq \frac{(\int_X f d\mu)^2}{\int_X f^2 d\mu}$.

(2) Show that no non-trivial lower bound for $\mu\{f > 0\}$ can be obtained in terms of $\int_X f d\mu$ alone.

Problem 60. Let μ be a finite measure. Then show that the following are equivalent.

(1) $f \in L^\infty(\mu)$.

(2) $f \in L^p(\mu)$ for all $p < \infty$ and $\sup\limits_p \|f\|_p < \infty$.

If these equivalent conditions hold, show that $\|f\|_p \to \|f\|_\infty$.

Problem 61. (1) If $f, g \in L^2$, show that $\|f - g\|^2 + \|f + g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

(2) (*) If $f_1, \ldots, f_n \in L^2(\mu)$, show that $(\langle f_i, f_j \rangle_{L^2(\mu)})_{1 \leq i,j \leq n}$ is a positive semi-definite matrix.

(3) Give examples to show that $L^p(X, \mathcal{F}, \mu)$ is not a Hilbert space if $p \neq 2$.

Problem 62. Let μ be a probability measure on X. If f, g are non-negative measurable and $fg \geq 1$ a.e.[μ], then $\left(\int_X f d\mu\right) \left(\int_X g d\mu\right) \geq 1$.

Problem 63. Suppose μ_n is a sequence of Radon measures on $\mathbb{R}^d$ such that $\lim_{n\to\infty} \int f d\mu_n$ exists and is finite for every $f \in C_c(\mathbb{R}^d)$. Then show that there is a Radon measure μ such that $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^d)$.

Problem 64. Let $1 \leq p \leq \infty$. Is $L^p(\mathbb{R}, \mathcal{B}, \lambda_1)$ separable in L^p metric?

Problem 65. Suppose $f_n \in L^1(\mu) \cap L^2(\mu)$. If $f_n \to g$ in $L^1(\mu)$ and $f_n \to h$ in $L^2(\mu)$, then $g = h$ a.e.$[\mu]$.

Problem 66. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces such that $\nu = \mu \circ T^{-1}$ for some $T : X \mapsto Y$. Then show that for any $f \in L^1(\nu)$, the function $f \circ T \in L^1(\mu)$ and that $\int_X (f \circ T) d\mu = \int_Y f d\nu$.

Problem 67. Let L be a positive linear functional on $C_c^\infty(\mathbb{R}^d)$ (endowed with sup-norm). Show that $L(f) = \int_X f d\mu$ for a unique Radon measure μ on $\mathbb{R}^d$.

Problem 68. Which of the following sets is dense in $L^p([0,1], \mathcal{B}, \lambda_1)$? Consider the case $p = \infty$ carefully.

(1) $C[0,1]$.

(2) $C^\infty[0,1]$.

(3) The set of all polynomials.

(4) The collection of all step functions.

Problem 69. If μ is a finite measure, show that $\frac{\|f\|_{p+1}^{p+1}}{\|f\|_p^p} \to \|f\|_\infty$ as $p \to \infty$.

Problem 70. Suppose $0 < p < 1$. If f, g are non-negative measurable function on $(X, \mathcal{F}, \mu)$, then $\left(\int_X (f+g)^p\right)^{1/p} \geq \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}$.

Problem 71. Let D be a bounded open set in the complex plane and let μ be the Lebesgue measure on D. Let H be the collection of all holomorphic functions on $\mathbb{D}$ that are in $L^2(D)$. Show that H is a closed subspace in L^2 norm and hence a Hilbert space itself.

Problem 72. Suppose $1 \leq p < q < r < \infty$. If $f_n \in L^p(\mu) \cap L^r(\mu)$ and this sequence is Cauchy in $L^p(\mu)$ and Cauchy in $L^r(\mu)$, then f_n converges in $L^q(\mu)$.

Problem 73. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$. Write out explicitly the meaning of $\mathbf{1}_{A_n} \to 0$ in $a.e.[\mu]$ sense, in measure and in L^1. Which of these imply the others? (Do this directly, without invoking the general theorems proved in class). What if μ is finite?

Problem 74. Let μ be a measure that is not supported at a single point. Show that $L^p(\mu)$ norm does not come from an inner product if $p \neq 2$ and does come from an inner product if $p = 2$.

Problem 75. Suppose μ is a Borel measure on $[0, 1]$ such that $\int fgd\mu = \left(\int fd\mu\right)\left(\int gd\mu\right)$ for all $f, g \in C[0, 1]$. Show that $\mu = \delta_a$ for some $a \in [0, 1]$.

Problem 76. Suppose f_n, f are non-negative measurable functions such that $f_n \to f$ $a.e.[\mu]$. Show that $\int_X f_n d\mu \to \int_X f d\mu$ if and only if $\int_X |f_n - f| d\mu \to 0$.

Problem 77. If $f_n \to f$ in measure μ, and $|f_n| \leq g$ for some integrable function g, then show that $\int_X |f_n - f| d\mu \to 0$ (DCT under convergence in measure only).

Problem 78. Let f_n, f be measurable functions on $(X, \mathcal{F}, \mu)$. If $\mathbf{1}_{|f_n - f| \geq \delta} \to 0$ a.e.$[\mu]$ for every $\delta > 0$, then is it true that $f_n \to f$ a.e.$[\mu]$?

Problem 79. (**16 marks**) State True or False and justify accordingly.

(1) If $f \in L^1(\mu) \cap L^3(\mu)$, then $f \in L^2(\mu)$.

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \mapsto \mathbb{R}$ be an integrable function. Let $A_n \in \mathcal{F}$ be a sequence of measurable sets. If $\mu(A_n) \to 0$ as $n \to \infty$, then $\int_{A_n} f d\mu \to 0$.

(3) On a measure space $(X, \mathcal{F}, \mu)$ suppose $f_n \to f$ $a.e.[\mu]$ and $\int f_n d\mu \to \int f d\mu$. Then there is an integrable function g such that $|f_n| \leq g$ $a.e.[\mu]$, for all n.

(4) Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined as $f(x, y) = x - y$ and let $\mathcal{F}$ be the smallest σ-algebra on $\mathbb{R}^2$ such that f is Borel measurable. Then $\mathcal{F}$ does not contain any non-empty bounded set in the plane.

Problem 80. (**10 marks**) If $f : \mathbb{R} \mapsto \mathbb{R}$ is right-continuous, then f is Borel measurable.

Problem 81. (**10 marks**) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \mapsto \mathbb{R}_+$ be a non-negative measurable function. Define $\theta(A) = \int_A f d\mu$ for $A \in \mathcal{F}$.

(1) Show that θ is a measure on $\mathcal{F}$.

(2) Show that $g : X \mapsto \mathbb{R}$ is integrable w.r.t. θ if and only if gf is integrable w.r.t. μ and $\int_X g d\theta = \int_X g f d\mu$.

Problem 82. (**10 marks**) Suppose $1 < p_1 < p_2 < p_3 < \infty$ are such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. If f_1, f_2, f_3 are measurable functions on $(X, \mathcal{F}, \mu)$ such that $f_i \in L^{p_i}(\mu)$ for $i = 1, 2, 3$, then show that $f_1 f_2 f_3 \in L^1$ and $\|f_1 f_2 f_3\|_1 \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3}$.

Problem 83. (**10 marks**) Let $(X, \mathcal{F}, \mu)$ be a measure space and let f be a non-negative measurable function.

(1) If f takes only integer values, then show that $\int f d\mu = \sum_{n=1}^{\infty} \mu\{f \geq n\}$.

(2) In general (i.e., for non-negative measurable f), show that f is integrable if and only if $\sum_{n=1}^{\infty} \mu\{f \geq n\}$ converges.

Problem 84. If f is a measurable function on $(X, \mathcal{F}, \mu)$ such that $\int f d\mu = \int f^2 d\mu = \int f^4 d\mu$ (all integrals assumed to exist and finite). Then $f = \mathbf{1}_A$ for some $A \in \mathcal{F}$.

Problem 85. If f is a non-negative measurable function such that $\log f$ is integrable, then

(1) $\int \frac{f^p - 1}{p} d\mu \to \int \log f d\mu$ as $p \downarrow 0$.

(2) $\frac{1}{p} \log \left(\int f^p d\mu \right) \to \int \log f d\mu$ as $p \downarrow 0$.

Problem 86. Suppose $f_n \to f$ $a.e.[\mu]$ and that $\sup\limits_n \int f_n^2 d\mu < \infty$. Then $\int |f_n - f| d\mu \to 0$.

Problem 87. If $f_n \to f$ $a.e.[\mu]$ in $(X, \mathcal{F}, \mu)$, then show that there exist $a_n \uparrow \infty$ such that $a_n X_n \to 0$ $a.s.[\mu]$.

Problem 88. Let X and Y be separable metric spaces and let $Z = X \times Y$ endowed with the product topology.

(1) If X and Y are separable, show that $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_Z$.

(2) Suppose $X = Y$ is not separable. Then show that $D = \{(x, x) : x \in X\}$ is in $\mathcal{B}_Z$ but not in $\mathcal{B}_X \times \mathcal{B}_Y$.

Problem 89. Let μ be a Radon measure on $\mathbb{R}$. Suppose convergence in measure w.r.t. μ implies convergence almost everywhere w.r.t. μ. What can you say about μ?

Problem 90. Let $\mathcal{L}_d$ denote the Lebesgue sigma algebra on $\mathbb{R}^d$. Show that $\mathcal{L}_2 \neq \mathcal{L}_1 \times \mathcal{L}_1$.

Problem 91. let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $f(x,y) = \sin(x)\mathbf{1}_{y<x<y+2\pi}$. Show that $\iint f(x,y)dydx \neq \iint f(x,y)dxdy$. Does this indicate a fatal flaw in Fubini's theorem as presented in thousands of books?

Problem 92. Let A be a Borel set in $\mathbb{R}^2$ such that its intersection with each vertical line is a finite set. Show that for $a.e.\ y[\lambda_1]$, the intersection of A with the horizontal line through $(0,y)$ has zero Lebesgue measure (in one dimension).

Problem 93. Suppose A, B are measurable subsets of $[0,1)$ (treated as a group with addition modulo 1). Show that there exists some $x \in [0,1)$ such that $\lambda(A \cap (B+x)) \geq \lambda(A)\lambda(B)$.

Problem 94. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a non-negative measurable function. Show that $\int_{\mathbb{R}} f(x)d\lambda(x)$ is equal to the area (two-dimensional Lebesgue measure) of $\{(x,y) : 0 \leq y \leq f(x)\}$ (the region between the graph of f and the x-axis.).

Problem 95. If f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$. Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\}dt$. [*Hint:* Use Fubini's theorem on $X \times \mathbb{R}_+$ with ...]

Problem 96. Let $A \subseteq \mathbb{R}$ be a measurable set with $\lambda_1(A) = 1$. Find $\lambda_2\{(x,y) \in \mathbb{R}^2 : (1+x^2)(y-e^x) \in A\}$.

Problem 97. Let f_n be measurable functions on $(X, \mathcal{F}, \mu)$ such that $f = \sum_n f_n$ converges $a.e.[\mu]$. From Fubini's and Tonelli's theorem discuss when the identity $\int f d\mu = \sum_n \int f_n d\mu$ holds. Consider the case when f_n are non-negative separately. How does this compare to the conditions required by MCT and DCT?

Problem 98. If $(X_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2, 3$, are σ-finite measure spaces. Show that

$$(\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 = (\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 \quad \text{and} \quad (\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3).$$

This justifies writing $\mu_1 \times \mu_2 \times \mu_3$ etc.

Problem 99. Write $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^2 \setminus \{0\} = (0,\infty) \times [0, 2\pi)$ with the identification $(x,y) \leftrightarrow (r,\theta)$ if $x = r\cos\theta$ and $y = r\sin\theta$ (polar co-ordinates). Let λ denote the one-dimensional Lebesgue measure on $\mathbb{R}$. Let λ_+ and λ_0 denote its restriction to $(0,\infty)$ and to $[0, 2\pi)$.

(1) Show that on $\mathbb{R}^2$ (or $\mathbb{R}^2 \setminus \{0\}$), the two measures $\lambda \times \lambda$ and $\lambda_+ \times \lambda_0$ are equal.

(2) Deduce the integration formula $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\ dx\ dy = \int_{(0,\infty)}\int_{[0,2\pi)} f(r\cos\theta, r\sin\theta) r d\theta dr$ for $f \in L^1(\lambda_2)$.

Problem 100. Let μ, ν be Borel probability measures on $\mathbb{R}$. Show that $(x+y)^2$ is integrable w.r.t. $\mu \times \nu$ if and only if x^2 is integrable w.r.t. μ and w.r.t. ν.

Problem 101. Let μ_n be a sequence of regular Borel measures on $\mathbb{R}$ such that $\lim_{n\to\infty} f d\mu_n$ exists and is finite for every $f \in C_c(\mathbb{R})$. Show that there is a regular Borel measure μ on $\mathbb{R}^d$ such that $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^d)$.

Problem 102. Let μ_n, μ be regular Borel measures on $\mathbb{R}$ such that $\lim_{n\to\infty} f d\mu_n = \int f d\mu$ for every $f \in C_c(\mathbb{R})$. Then show that $\mu_n(a,b) \to \mu(a,b)$ for all except atmost countably many a, b.

Problem 103. Show that there exists probability measures μ_n, ν_n on $[0,1]$ such that $\int f d\mu_n - \int f d\nu_n \to 0$ as $n \to \infty$, for any $f \in C[0,1]$ but such that $\mu_n[0,t] - \nu_n[0,t] \not\to 0$ for any $t \in (0,1)$.

Problem 104. Let μ, ν be finite measures on $(X, \mathcal{F})$. Show that $\nu \ll \mu$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\nu(A) < \epsilon$ for any $A \in \mathcal{F}$ with $\mu(A) < \delta$.

Problem 105. Construct a sigma-finite measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu \ll \lambda_1$ and $\mu(a,b) = \infty$ for any $a < b$.

Problem 106. Let $(X, \mathcal{F}, \mu)$ be a probability space. Let $A_n \in \mathcal{F}$ and let $A = \limsup A_n$.

(1) If $\sum_n \mu(A_n) < \infty$, then show that $\mu(A) = 0$.

(2) If $\mu(A_n) \geq \epsilon > 0$ for all n, then show that $\mu(A) \geq \epsilon$.

Problem 107. Let $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a non-negative integrable function. Show that there exists a radial function $g : \mathbb{R}^d \mapsto \mathbb{R}$ (this means that $g(x) = g(y)$ if $|x| = |y|$) such that $\lambda_d\{f > t\} = \lambda_d\{g > t\}$ for all $t > 0$. Deduce that $\int f d\lambda_d = \int g d\lambda_d$. [*Note:* g is called the spherical rearrangement of f.]

Problem 108. On $(X, \mathcal{F}, \mu)$, suppose f is a non-negative integrable function. Let

$$I(\lambda) = \sum_{n\in\mathbb{Z}} \lambda^n \mu\{\lambda^n \leq f < \lambda^{n+1}\}.$$

Show that $I(\lambda)$ is well-defined (i.e., the series converges absolutely) for every $\lambda > 1$ and that $I(\lambda) \downarrow \int f d\mu$ as $\lambda \downarrow 1$.

Problem 109. Let f be a non-negative measurable function on the sigma-finite measure space $(X, \mathcal{F})$ and let $\nu(A) = \int_A f d\mu$. Show that ν is sigma-finite.

Problem 110. Let f be a non-negative measurable function on the sigma-finite measure space $(X, \mathcal{F})$. If $\mathcal{G}$ is a sub-sigma algebra of $\mathcal{F}$, then show that there exists a $g : X \mapsto \mathbb{R}_+$ that is Borel measurable w.r.t. $\mathcal{G}$ and such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{G}$.

Problem 111. Suppose $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ (all sigma-finite measure on some measure spaces). Then show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

Problem 112. (**16 marks**) State "True" or "False" and justify accordingly.

(1) Let $\mathcal{F}$ be the collection of all those Borel subsets of $\mathbb{R}$ that have zero Lebesgue measure or whose complement has zero Lebesgue measure. Then $\mathcal{F}$ is a σ-algebra.

(2) If f is an integrable function on $(X, \mathcal{F}, \mu)$, then $t\mu\{f > t\} \to 0$ as $t \to \infty$.

(3) Suppose A_n are subsets of $[0, 1]$ such that $\lambda(A_n) \leq \frac{1}{2}$ for all n. Then $\lambda(\limsup A_n) \leq \frac{1}{2}$.

(4) Suppose μ and ν are measures on $(X, \mathcal{F})$ such that ν is absolutely continuous to μ. If μ is a finite measure, then so is ν.

(5) For $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let $A_k = \{k, 2k, 3k, \ldots\}$. Then the sigma algebra generated by the sets A_p as p varies over all primes, is the power set of $\mathbb{N}$.

Problem 113. (**16 marks**)

(1) Let $\mathcal{F}$ be the sigma-algebra on $\mathbb{R}$ generated by the intervals $(0, 2)$ and $(1, 3)$. What is the cardinality of $\mathcal{F}$?

(2) Let μ^* be an outer measure on$X$. Suppose $A_1, B_1, A_2, B_2, \ldots$ are subsets of X such that $\mu^*(A_n \Delta B_n) = 0$ for each $n \geq 1$. Show that $\mu^*(\cup_n A_n) = \mu^*(\cup_n B_n)$.

(3) Suppose $A_1, \ldots, A_9$ are Borel subsets of $[0, 1]$ such that each $x \in [0, 1]$ belongs to at least three distinct sets among $A_1, \ldots, A_9$. Show that $\lambda(A_k) \geq \frac{1}{3}$ for some k.

(4) Let $A \subseteq \mathbb{R}$ be a Borel set with $\lambda_1(A) = 1$. Find $\lambda_2\{(x, y) \in \mathbb{R}^2 : (1 + x^2)(y - e^x) \in A\}$.

Problem 114. (**6 marks**) Suppose f_n, f, g are measurable functions on $(X, \mathcal{F}, \mu)$.

(1) If $\mathbf{1}_{|f_n - f| \geq \delta} \to 0$ a.e.$[\mu]$ for every $\delta > 0$, then show that $f_n \to f$ a.e.$[\mu]$.

(2) Assume that f and g are integrable. If $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e.$[\mu]$.

Problem 115. (**6 marks**) Given $f : \mathbb{R}^2 \mapsto \mathbb{R}$, define $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(x) = f(x, 1 - x)$.

(1) Take the Borel sigma algebras on $\mathbb{R}$ and $\mathbb{R}^2$. If f is measurable, then so is g.

(2) Take the Lebesgue sigma algebras on $\mathbb{R}$ and $\mathbb{R}^2$. Show that g need not be measurable even if f is measurable.

Problem 116. (**6 marks**) Consider $\mathbb{R}$ with the Borel sigma algebra.

(1) Let $f, g : [0,1] \mapsto \mathbb{R}$ be Borel measurable functions. If $h(x,y) = f(x) + g(y)$ is integrable over $[0,1]^2$, then f and g are integrable over $[0,1]$.

(2) If $f : \mathbb{R} \mapsto \mathbb{R}$ is integrable, then $\sum\limits_{n\in\mathbb{Z}} f(x+n)$ converges for a.e. x (w.r.t. Lebesgue measure).

Problem 117. (**5 marks**) If $f : \mathbb{R} \mapsto \mathbb{R}$ is integrable and $g(x) = \int\limits_{[x,x+1]} f d\lambda$, then show that g is a continuous function.

Problem 118. (**5 marks**) Suppose μ is a Borel measure on $\mathbb{R}$ such that $\mu(A) = 1$ whenever A is a Borel set with $\lambda(A) = 1$. Show that $\mu = \lambda$.