PROBLEM SET 1
(MEASURE THEORY)

TO BE DISCUSSED ON 24TH JANUARY IN TUTORIALS

Problem 1. In each of the following cases, find $\sigma(S)$ and $\mathcal{A}(S)$ (the sigma-algebra and algebra generated by S).

(1) X is a set and S is the collection of all singleton subsets of X.

(2) X is a set and S is the collection of all two-element subsets of X.

(3) $A_1, A_2, \ldots$ are pairwise disjoint sets of X such that $\bigcup_n A_n = X$.

(4) Do the previous exercise if A_i are pairwise disjoint but their union may not equal X.

Problem 2. Let $\mathcal{F}$ be a sigma algebra on X. Let $A_1, A_2, \ldots$ be elements of $\mathcal{F}$. Show that the following sets are also in $\mathcal{F}$ (first express the set in proper notation).

(1) The set of $x \in X$ that belong to exactly five of the A_ns.

(2) The set of $x \in X$ that belong to all except five of the A_ns.

(3) The set of $x \in X$ that belong to infinitely many of the A_ns.

(4) The set of $x \in X$ that belong to all but finitely many of the A_ns.

Problem 3. Decide whether the following statements are true or false and justify your answer.

(1) A finite union of σ-algebras is not necessarily a σ-algebra.

(2) If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ is an increasing sequence of sigma-algebras of a set X, then $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is also a sigma-algebra.

(3) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : X \mapsto Y$ be a function. Then $\mathcal{G} := \{T(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y.

(4) Let $\mathcal{F}$ be a sigma-algebra on X and let $T : Y \mapsto X$ be a function. Then $\mathcal{G} := \{T^{-1}(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y. Here $T^{-1}(A) = \{y \in Y : T(y) \in A\}$.

(5) There is no sigma-algebra with exactly 1000 elements.

Problem 4. Let $\mathcal{F} = \sigma(S)$ where S is a collection of subsets of X. Suppose $a, b \in X$ are such that every set in S either contains both a and b or does not contain either. Then show that the same property holds for sets in $\mathcal{F}$.

Problem 5. Consider the outer Lebesgue measure λ_2^* on $\mathbb{R}^2$. Show that $[a, b] \times [c, d]$ in Lebesgue measurable by checking the Carathéodary cut condition.

Problem 6. Suppose $A \subseteq B$ are (Lebesgue) measurable subsets of R and $\lambda(A) = \lambda(B)$. Then show that any set C such that $A \subseteq C \subseteq B$ is also measurable and that $\lambda(C) = \lambda(A)$.

Problem 7. (*) Let (X, d) be a compact metric space. Fix $\alpha > 0$ and define

$$\mu_\alpha^*(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^{\infty} \text{dia}(B_j)^\alpha : B_j \text{ are open balls in } X \text{ such that } \text{dia}(B_j) < \delta \text{ and } \bigcup_j B_j \supseteq A \right\}.$$

Show that μ_α^* is an outer measure.

When $X = \mathbb{R}$ with the usual metric, μ_α^* is the same as λ^* (the Lebesgue outer measure) when $\alpha = 1$. Further, μ_α^* is zero if $\alpha > 1$ and $\mu_\alpha^*(A) = \infty$ if $\alpha < 1$ and A is any interval.