PROBLEM SET 10
(MEASURE THEORY)

Problem 1. Let X and Y be separable metric spaces and let $Z = X \times Y$ endowed with the product topology.

(1) If X and Y are separable, show that $\mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_Z$.

(2) Suppose $X = Y$ is not separable. Then show that $D = \{(x, x) : x \in X\}$ is in $\mathcal{B}_Z$ but not in $\mathcal{B}_X \times \mathcal{B}_Y$.

Problem 2. Let $\mathcal{L}_d$ denote the Lebesgue sigma algebra on $\mathbb{R}^d$. Show that $\mathcal{L}_2 \neq \mathcal{L}_1 \times \mathcal{L}_1$.

Problem 3. let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $f(x, y) = \sin(x)\mathbf{1}_{y<x<y+2\pi}$. Show that $\iint f(x, y)dydx \neq \iint f(x, y)dxdy$. Does this indicate a fatal flaw in Fubini's theorem as presented in thousands of books?

Problem 4. Let A be a Borel set in $\mathbb{R}^2$ such that its intersection with each vertical line is a finite set. Show that for $a.e.\ y[\lambda_1]$, the intersection of A with the horizontal line through $(0, y)$ has zero Lebesgue measure (in one dimension).

Problem 5. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a non-negative measurable function. Show that $\int_{\mathbb{R}} f(x)d\lambda(x)$ is equal to the area (two-dimensional Lebesgue measure) of $\{(x, y) : 0 \leq y \leq f(x)\}$ (the region between the graph of f and the x-axis.).

Problem 6. If f be a non-negative measurable function on $(X, \mathcal{F}, \mu)$. Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\}dt$. [*Hint:* Use Fubini's theorem on $X \times \mathbb{R}_+$ with ...]

Problem 7. If $(X_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2, 3$, are σ-finite measure spaces. Show that

$$(\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 = (\mathcal{F}_1 \times \mathcal{F}_2) \times \mathcal{F}_3 \quad \text{and} \quad (\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3).$$

This justifies writing $\mu_1 \times \mu_2 \times \mu_3$ etc.