PROBLEM SET 2
(MEASURE THEORY)

TO BE DISCUSSED ON 31ST JANUARY IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(1) If $A_n, A \in \mathcal{F}$ and $A_n \uparrow A$, show that $\mu(A_n) \uparrow \mu(A)$.

(2) If $A_n, A \in \mathcal{F}$ and $A_n \downarrow A$ and $\mu(A_n) < \infty$ for some n, then show that $\mu(A_n) \downarrow \mu(A)$.

(3) Show that the second conclusion may fail if $\mu(A_n) = \infty$ for all n.

Problem 2. A measure μ on $(X, \mathcal{F})$ is said to be σ-finite if there exist $E_1, E_2, \ldots$ in $\mathcal{F}$ such that $X = \bigcup_n E_n$ and $\mu(E_n) < \infty$ for all n.

(1) Show that a σ-finite measure space has sets of arbitrarily high but finite measure.

(2) Show that a σ-finite measure has at most countably many atoms. Show that the previous assertion is false without the σ-finiteness assumption.

Problem 3. Let $\mathcal{F} = \sigma(S)$ be a sigma algebra on X. Show that for any $A \in \mathcal{F}$, there exists countably many sets $A_1, A_2, \ldots$ in S such that $A \in \sigma(\{A_1, A_2, \ldots\})$.

Problem 4. Let $\mathcal{F}$ be a sigma algebra on X and assume that $B \subseteq X$ is not in $\mathcal{F}$. Show that the smallest sigma algebra containing $\mathcal{F}$ and B is the collection of all sets of the form $(A_1 \cap B) \cup (A_2 \cap B^c)$ where $A_1, A_2 \in \mathcal{F}$.

(*) If μ is a measure on $\mathcal{F}$, can you extend it to $\mathcal{G}$ in some way? Is the extension unique?

Problem 5. (*) Show that any convex set in $\mathbb{R}^d$ is (Lebesgue) measurable. Is it necessarily Borel measurable?

Problem 6. (*) If $A \subseteq \mathbb{R}$ is measurable and has positive Lebesgue measure, show that it contains a three-term arithmetic progression, i.e., there exist $a, b \in A$ such that $\frac{1}{2}(a+b) \in A$.

Problem 7. Let $\mathcal{A}' \subseteq \mathcal{A}$ be two algebras on X that genberate the same sigma algebra (call it $\mathcal{F}$). Now suppose we have a countably additive measure μ on $\mathcal{A}$ and let μ' be the restriction of μ to $\mathcal{A}'$.

By the theorem proved in class, both μ' and mu extend to $\mathcal{F}$ as measures. Are the Carathéodary sigma algebras the same? If yes, are the extended measures equal? On the way, is the outer measure constructed from μ and μ' the same?

As a particular case, if we start with a measure μ on a sigma algebra $\mathcal{F}$, and extend it (since $\mathcal{F}$ is also an algebra), what is the extended sigma algebra? Is it $\mathcal{F}$ or is it larger?

Problem 8. Let $X = \{0,1\}^{\mathbb{N}}$ be the sequence space of zeros and ones. An element $\omega \in X$ is written as $\omega = (\omega_1, \omega_2, \ldots)$.

(1) A cylinder set is one defined by specifying the values of finitely many co-ordinates. Eg., $\{\omega : \omega_1 = 0, \omega_2 = 1, \omega_7 = 1\}$. Show that the complement of a cylinder set is a finite union of pairwise disjoint cylinder sets. Use this to describe $\mathcal{A}(S)$ as the collection of all finite unions of pairwise disjoint cylinder sets.

(2) If A is a cylinder set for which exactly n co-ordinate values are specified, define $\mu_0(A) = 2^{-n}$. Extend in the obvious way to $\mathcal{A}(S)$ and show that mu_0 is countably additive on $\mathcal{A}(S)$.

(3) Argue that there is a measure μ on $\sigma(S)$ that extends μ_0.

[*Note:* This exercise is to make precise the notion of a infinite sequence of fair coin tosses.]