PROBLEM SET 3
(MEASURE THEORY)

TO BE DISCUSSED ON 7TH FEBRUARY IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Let A and B be measurable subsets of the line.

(1) Show that $\lambda(A+B) \geq \lambda(A) + \lambda(B)$. [*Note:* The discrete version of this is: If A and B are subsets of $\mathbb{Z}$, then $|A+B| \geq |A| + |B| - 1$, where $|A|$ denotes the cardinality of A.]

(2) Show that there can be no reverse inequality by constructing A, B having zero measure but such that $A + B = \mathbb{R}$.

Problem 2. Let A be a bounded subset of $\mathbb{R}$ with positive Lebesgue measure. Given are $\epsilon > 0$ and a sequence δ_n converging to zero. Show that there exists $B \subseteq A$ and a subseequnce δ_{n_k} such that (a) B is measurable and $\lambda(B) > \lambda(A) - \epsilon$, (b) if $x \in B$ then $x \pm \delta_{n_k} \in B$ for all k.

Problem 3. Let A be a measurable subset of $\mathbb{R}^2$ with positive Lebesgue measure. Show that there is a measurable set B having positive measure and a number $\delta > 0$ such that $B, B + (\delta, 0), B + (0, \delta), B + (\delta, \delta)$ are all subsets of A.

Problem 4. If $A \subseteq \mathbb{R}^2$ is measurable, then $\lambda_2(A) = \inf \sum_n |B_n|$ where the infimum is over all countable coverings of A by open balls $B_1, B_2, \ldots$ and $|B_n|$ is the area. Deduce that Lebesgue measure on $\mathbb{R}^2$ is rotation-invariant.

[*Note:* We defined Lebesgue measure using coverings by open rectangles (with sides parallel to the axes). This exercise is to show that the shape of the basic sets does not affect the measure we get].

Problem 5. True or false?

(1) If $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is continuous and $A \subseteq \mathbb{R}^2$ is measurable, then $f(A) := \{f(x) : x \in A\}$ is also measurable.

(2) Let $\mu : \mathcal{A} \mapsto [0, \infty]$ be a finitely additive set function on an algebra $\mathcal{A}$. Then countable additivity of μ is equivalent to continuity of μ under monotone limits (i.e., $A_n \uparrow A$, $A_, A \in \mathcal{A}$ implies $\mu(A_n) \uparrow \mu(A)$ and similarly for decreasing limits).

(3) $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on X. If μ is a σ-finite measure on $\mathcal{F}$ then its restriction to $\mathcal{G}$ is also σ-finite.