PROBLEM SET 3
(MEASURE THEORY)

TO BE DISCUSSED ON 7TH MARCH IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. Show that any of the following conditions implies that f is Borel measurable.

(1) f is increasing.

(2) f is right-continuous.

(3) f is lower semi-continuous (means that $f(x) = \liminf_{y \to x} f(y)$ for all x).

Problem 2. Let $(X, \mathcal{F})$ be a measure space and let $f_n : X \mapsto \mathbb{R}$ be a sequence of Borel measurable functions.

(1) Show that $\sup_n f_n$, $\limsup f_n$, $\lim f_n$ (assuming limit exists), $\sum_n f_n$ (assuming that the sum converges pointwise), are measurable (where necessary, these functions may be allowed to take values $\pm\infty$ also).

(2) Show that the set $\{x \in X : \lim_n f_n(x) \text{ exists }\}$ is measurable in X. Same for $\{x : \limsup_n f_n(x) = +\infty\}$, $\{x : \lim f_n(x) = +\infty\}$.

(3) Show that the supremum of an uncountable family of measurable functions need not be measurable (even if the supremum is finite everywhere).

Problem 3. Let (X, Σ) be a measure space and suppose $f_1, f_2 : X \mapsto \mathbb{R}$. Then $F = (f_1, f_2) : X \mapsto \mathbb{R}^2$.

(1) Show that F is Borel measurable if and only if both f_1, f_2 are Borel measurable.

(2) Deduce that if f_1, f_2 are measurable, then so are $af_1 + bf_2$ (for any $a, b \in \mathbb{R}$) and $f_1 f_2$ and f_1/f_2 (for the last one, assume that $f_2 \geq 0$ so that f_1/f_2 can be defined unambiguously as an $\overline{\mathbb{R}}$-valued function).

(3) If $f : \mathbb{R} \mapsto \mathbb{R}$ is pointwise differentiable, then show that $f' : \mathbb{R} \mapsto \mathbb{R}$ is measurable.

Problem 4. Find a measurable function on an appropriate interval in $\mathbb{R}$ so that the push-forward of Lebesgue measure of the interval is the measure μ satisfying

(1) $\mu(a, b] = \int_a^b e^{-|x|} dx$ for $a < b$.

(2) $\mu(a, b] = \int_a^b \frac{1}{1+x^2} dx$ for $a < b$.

(3) $\mu(a,b]$ = number of integers in $(a,b]$.

Problem 5. Let K be the $1/3$-Cantor set.

(1) Write out in detail the sketch given in class that there is a bijection $T : [0,1] \mapsto K$ such that T and T^{-1} are Borel measurable.

(2) Use T to construct a Lebesgue measurable set in $\mathbb{R}$ that is not Borel measurable. [*Hint:* Use the existence of non-measurable sets.]

Problem 6. (*) This exercise is to show the isomorphism between $((0,1), \mathcal{B}_{(0,1)}, \lambda_1)$ and $((0,1)^2, \mathcal{B}_{(0,1)^2}, \lambda_2)$. As pointed out in class, the essential idea is to take a number $x = 0.x_1x_2...$ in binary expansion and map it to (y,z) where $y = 0.x_1x_3...$ and $z = 0.x_2x_4....$ But because of the ambiguities of binary expansion at dyadic rationals, this is not quite a bijection between $(0,1)$ and $(0,1)^2$. Here is how to fix this.

(1) Find $T_1 : (0,1) \mapsto (0,1)^2$ that is injective and such that $\mathrm{Im}(T_1)$ is a Borel set in $(0,1)^2$.

(2) Find $T_2 : (0,1)^2 \mapsto (0,1)$ that is injective and such that $\mathrm{Im}(T_2)$ is a Borel set in $(0,1)$.

(3) Use the idea of the proof of Schroder-Bernstein theorem to get a bijection $T : (0,1) \mapsto (0,1)^2$ so that T, T^{-1} are Borel measurable.

(4) If you base your T_2 on the binary expansion idea above, it turns out that Lebesgue measure on the two spaces are pushed forward to each other by T and T^{-1}.