PROBLEM SET 5
(MEASURE THEORY)

TO BE DISCUSSED ON 14TH MARCH IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Suppose the set of discontinuity points of $f : \mathbb{R} \mapsto \mathbb{R}$ has has zero Lebesgue measure. Show that f is a measurable function (w.r.t the Lebesgue measure).

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a measure space. If f $f : X \mapsto \mathbb{R}$ is a bounded measurable function. Show that there exist simple functions s_n such that $s_n \to f$ uniformly on X.

Problem 3. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{S}_+$.

(1) Show that $t \mapsto \mu\{f > t\}$ is measurable from $[0, \infty]$ to $[0, \infty]$.
(2) Show that $\int_X f d\mu = \int_0^\infty \mu\{f > t\} dt$.
(3) What about $\int_0^\infty \mu\{f \geq t\} dt$?

Problem 4. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Define $\nu : \mathcal{F} \mapsto [0, \infty]$ be defined by $\nu(A) = \int_X f \mathbf{1}_A d\mu$. Show that ν is a measure.

Problem 5. Let f be a non-negative simple function on $(X, \mathcal{F}, \mu)$. Show that for any $t > 0$ we have $\mu\{f \geq t\} \leq \frac{1}{t} \int_X f d\mu$ (Markov's inequality).

Problem 6. Consider the space of Riemann integrable functions on $[0, 1]$ with (pseudo)-metric $d(f, g) = \int_0^1 |f - g|$. Show that this metric space is not complete.