PROBLEM SET 6
(MEASURE THEORY)

TO BE DISCUSSED ON 21ST MARCH IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Suppose f, g are integrable functions on a measure space $X, \mathcal{F}, \mu)$. Which of the following are necessarily integrable? (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g, (e) $f \vee g$, (f) $f \wedge g$.

Problem 2. Given an integrable function f on $(X, \mathcal{F}, \mu)$, show that there exist simple functions s_n such that $\int_X |f - s_n| d\mu \to 0$ as $n \to \infty$.

Problem 3.
(1) If f_n are non-negative measurable functions on $(X, \mathcal{F}, \mu)$ and $f = \sum_n f_n$, then show that $\int_X f d\mu = \sum_n \int_n f_n d\mu$.
(2) If g is a non-negative measurable function and $\nu(A) := \int_A g d\mu$ for $A \in \mathcal{F}$, (convention: By $\int_A f d\mu$ we just mean $\int_X f \mathbf{1}_A d\mu$), then show that ν is a measure on $\mathcal{F}$.

Problem 4. Suppose $f : (a, b) \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that (a) $\theta \mapsto f(\theta, x)$ is differentiable for each $x \in \mathbb{R}$, (b) $x \mapsto f(\theta, x)$ is measurable for each $\theta \in (a, b)$, (c) $|f(\theta_1, x) - f(\theta_2, x)| \le g(x)|\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in (a, b)$ and $x \in \mathbb{R}$ and that g is integrable over $\mathbb{R}$ (w.r.t. Lebesgue measure). Define $H(\theta) := \int_{\mathbb{R}} f(\theta, x) d\lambda(x)$.

Show that H is differentiable, that $\frac{d}{d\theta} f(\theta, x)$ is integrable over $\mathbb{R}$ for each θ, and that

$$\frac{d}{d\theta} H(\theta) = \int_{\mathbb{R}} \frac{d}{d\theta} f(\theta, x) d\lambda(x).$$

Problem 5. Assume that f_n are non-negative measurable functions that are bounded above by an integrable function g a.e.[μ] (i.e. $0 \le f_n \le g$ a.e.[μ]).
(1) Show that $\limsup_{n\to\infty} \int_X f_n d\mu \le \int_X (\limsup_{n\to\infty} f_n) d\mu$.
(2) If $f_n \downarrow f$ a.e.[μ], and $\int_X f_n d\mu$ is finite for some n, then show that $\int_X f_n d\mu \downarrow \int_X f d\mu$.

Problem 6. If f is an integrable function, show that $\int |f| \mathbf{1}_{|f|>n} d\mu \to 0$. More generally, if A_n is any sequence of events such that $\mu(A_n) \to 0$, then show that $\int_{A_n} |f| d\mu \to 0$ as $n \to \infty$.

Problem 7. If $f : [a, b] \mapsto \mathbb{R}$ is a continuous function, show that its Riemann integral is equal to its Lebesgue integral (w.r.t. Lebesgue measure on $[a, b]$). [*Note:* This is true more generally.]

Problem 8. Let A_n be measurable sets in $(X, \mathcal{F}, \mu)$. Let A be the set of all x that belong to A_n for infinitely many n (is A measurable?).

(1) If $\sum_n \mu(A_n) < \infty$, then show that $\mu(A) = 0$.

(2) Show that $\mu(A_n) \to 0$ does not necessarily imply that $\mu(A) = 0$.

Problem 9. If f_n are non-negative measurable functions with $\int_X f_n d\mu = 1$ for all n and $f_n \to f$ a.e.[μ], then show that the set of all possible values of $\int_X f d\mu$ is $[0, 1]$.