PROBLEM SET 8
(MEASURE THEORY)

TO BE DISCUSSED ON 4TH APRIL IN TUTORIALS. PROBLEMS MARKED (*) ARE OPTIONAL.

Problem 1. Let $1 \leq p \leq \infty$. Is $L^p(\mathbb{R}, \mathcal{B}, \lambda_1)$ separable in L^p metric?

Problem 2. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces such that $\nu = \mu \circ T^{-1}$ for some $T : X \mapsto Y$. Then show that for any $f \in L^1(\nu)$, the function $f \circ T \in L^1(\mu)$ and that $\int_X (f \circ T) d\mu = \int_Y f d\nu$.

Problem 3. Let L be a positive linear functional on $C_c^\infty(\mathbb{R}^d)$ (endowed with sup-norm). Show that $L(f) = \int_X f d\mu$ for a unique Radon measure μ on $\mathbb{R}^d$.

Problem 4. Which of the following sets is dense in $L^p([0,1], \mathcal{B}, \lambda_1)$? Consider the case $p = \infty$ carefully.

(1) $C[0,1]$.

(2) $C^\infty[0,1]$.

(3) The set of all polynomials.

(4) The collection of all step functions.

Problem 5. If μ is a finite measure, show that $\frac{\|f\|_{p+1}^{p+1}}{\|f\|_p^p} \to \|f\|_\infty$ as $p \to \infty$.

Problem 6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$. Write out explicitly the meaning of $\mathbf{1}_{A_n} \to 0$ in $a.e.[\mu]$ sense, in measure and in L^1. Which of these imply the others? (Do this directly, without invoking the general theorems proved in class). What if μ is finite?

Problem 7. Let μ be a measure that is not supported at a single point. Show that $L^p(\mu)$ norm does not come from an inner product if $p \neq 2$ and does come from an inner product if $p = 2$.

Problem 8. Suppose f_n, f are non-negative measurable functions such that $f_n \to f$ $a.e.[\mu]$. Show that $\int_X f_n d\mu \to \int_X f d\mu$ if and only if $\int_X |f_n - f| d\mu \to 0$.

Problem 9. if $f_n \to f$ in measure μ, and $|f_n| \leq g$ for some integrable function g, then show that $\int_X |f_n - f| d\mu \to 0$ (DCT under convergence in measure only).