PROBLEM SET: BROWNIAN MOTION

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Problem 1. Let $X \sim N_n(0, \Sigma)$. Show that

$$\mathbf{E}[X_1 \dots X_n] = \sum_M \prod_{i=1}^{n/2} \sigma_{M_{i,1}, M_{i,2}}$$

where the sum is over matchings $M = \{\{M_{1,1}, M_{1,2}\}, \{M_{2,1}, M_{2,2}\}, \dots\}$ of the set $\{1, 2, \dots, n\}$. For example, if $n = 4$, the possible matchings are $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$.

Problem 2. Let $Z \sim N(0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $|f(x)|, |f'(x)|$ are bounded by e^{ax^2} for some $a < \frac{1}{2}$.

(1) Show that $\mathbf{E}[Zf'(Z)] = \mathbf{E}[f(Z)]$.

(2) Deduce that $\mathbf{E}[Z^{2n}] = (2n-1) \times (2n-3) \times \dots \times 3 \times 1$.

Problem 3. If $X \sim N_n(0, A)$ and $Y \sim N_n(0, B)$ are independent, what is the distribution of $Z_\theta = \cos\theta X + \sin\theta Y$, where $0 \leq \theta \leq \frac{\pi}{2}$.

Problem 4. Let $Z \sim N(0,1)$. Show that for any $0 < p < 1$, the shortest interval $I = [a, b]$ such that $\mathbf{P}\{Z \in I\} = p$ is the one with $b = -a$.

Problem 5. For any interval $I = [a, b]$, let $V(I) = \int_a^b \varphi$ denote its "Gaussian volume" and let $S(I) = \varphi(a) + \varphi(b)$ (defined by taking limits if $a = -\infty$ or $b = +\infty$) denotes its "Gaussian surface area". Show that among all intervals with given volume $p \in (0,1)$, the one with the least surface area is of the form $(-\infty, b]$ or $[a, \infty)$.

Problem 6. Let $Z \sim N(0,1)$. Derive the tail bounds

$$ce^{-\frac{1}{2}t^2 - t} \leq \mathbf{P}\{Z \geq t\} \leq e^{-\frac{1}{2}t^2}$$

for some $c > 0$, as follows.

(1) Show that $\mathbf{E}[e^{\theta Z}] = e^{\theta^2/2}$ for $t \in \mathbb{R}$ (in fact for $t \in \mathbb{C}$) and hence deduce the upper bound.

Date: May 22, 2026.

(2) Show that the Radon-Nikodym derivative of $N(0,1)$ w.r.t. $N(t,1)$ is $e^{-xt+\frac{1}{2}t^2}$ and hence deduce that $\mathbf{P}\{Z \geq t\} \geq (\Phi(a) - \frac{1}{2})e^{-t(t+a)+\frac{1}{2}t^2}$ for any $t, a > 0$. Deduce the lower bound.

Problem 7. Let $\mu = \mu_1 \otimes \mu_2 \otimes \ldots$ be a product probability measure on $\mathbb{R}^{\mathbb{N}}$, where μ_k are probability measures on $\mathbb{R}$.

(1) For any $\varepsilon > 0$, show there is a compact $K \subseteq \mathbb{R}^{\mathbb{N}}$ such that $\mu(K) > 1 - \varepsilon$.

(2) If each μ_k gives strictly positive measure to each open subset of $\mathbb{R}$, show that the same is true of μ on $\mathbb{R}^{\mathbb{N}}$.

Problem 8. Show that there is no probability measure μ on $C[0,1]$ whose finite dimensional distributions are given by

$$\mu_{t_1,\ldots,t_n} = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \ldots \otimes \alpha_{t_n}$$

where α_t, $0 \leq t \leq 1$, are given *non-degenerate* probability distributions on $\mathbb{R}$.

Problem 9. Let $t_1, \ldots, t_n \geq 0$. Show that the following are equivalent.

(1) $W_1, \ldots, W_n$ are jointly Gaussian with zero means and $\mathbf{E}[W_i W_j] = t_i \wedge t_j$ for all i, j.

(2) $W_1, W_2 - W_1, W_3, W_2, \ldots, W_n - W_{n-1}$ are independent Gaussians with zero means and variances $t_1, t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}$, respectively.

Problem 10. Let $Z_1, \ldots, Z_n$ be i.i.d. standard Gaussians and let $M_n = \max\{Z_1, \ldots, Z_n\}$. Show that $\mathbf{P}\{M_n \geq (1+\varepsilon)\sqrt{2\log n}\} \leq n^{-\varepsilon}$ for any $\varepsilon > 0$. Deduce that $\limsup \frac{M_n}{\sqrt{2\log n}} \leq 1$ a.s.

Problem 11. Let W be standard Brownian motion. Let $s < t < u$.

(1) Find the conditional distribution of W_t given (W_s, W_u).

(2) Find the conditional distribution of (W_s, W_u) given W_t.

Problem 12. Suppose $X_1, \ldots, X_{2n+1}$ are jointly Gaussian with zero means and covariance $\mathbf{E}[X_i X_j] = t_i$ if $i \leq j$, where $0 \leq t_1 < \ldots < t_{2n+1}$ are fixed.

(1) Conditional on $X_1, X_3, \ldots, X_{2n+1}$, show that $X_2, X_4, \ldots, X_{2n}$ are independent Gaussians with

$$X_{2k} \sim N((1-\alpha_k)X_{2k-1} + \alpha_k X_{2k+1}, \alpha_k(1-\alpha_k))$$

where $\alpha_k = \frac{t_{2k}-t_{2k-1}}{t_{2k+1}-t_{2k-1}}$.

(2) Using the first part, describe how you would construct the random variables $W(k/2^m)$, $0 \leq k \leq 2^m$, and $m \geq 0$, from a sequence of i.i.d. standard Gaussian random variables. Here W is Brownian motion.

[*Remark:* In class we used general facts about Gaussians to construct $W(t)$, $t \in D$, a dense countable subset of $[0,1]$. This exercise gives an easier construction, specific to the particular covariance structure of Brownian motion]

Problem 13. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a positive definite kernel (meaning that $(K(t_i, t_j))_{i,j \le n}$ is symmetric and positive definite for any $n \ge 1$ and any $0 \le t_1 < \ldots < t_n \le 1$).

(1) Show that $d(t,s) := \sqrt{K(t,t) + K(s,s) - 2K(t,s)}$ is a metric on $[0,1]$.

(2) If X is a Gaussian process with mean zero and covariance K, what does the distance d mean in terms of X?

[*Note:* Positive semi-definiteness of K is sufficient for this problem, except that d is then a pseudo metric, not a metric.]

Problem 14. Let K and d be as in the previous problem. If $d(t,s) \le C|t-s|^{\varepsilon}$ for some $C < \infty$ and $\varepsilon > 0$, show that there exists a Gaussian process $X = (X_t)_{t \in [0,1]}$ with zero mean, covariance K and such that $t \mapsto X_t$ is continuous a.s. (Use results proved in class).

Problem 15. If W is standard Brownian motion, show that for any $0 < t_1 < \ldots < t_n$,

$$\mathbf{P}\{W_{t_1} < W_{t_2} < \ldots < W_{t_n}\} = \frac{1}{2^n}.$$

Deduce that almost surely, there is no interval of positive length on which W is increasing.

Problem 16. Let W be standard Brownian motion and let $V \sim \text{Exp}(1)$ be independent of W. Define $X(t) = W(t)$ for $t \neq V$ and $X(V) = 0$. Show that X and W have the same finite dimensional distributions. But X is not a Brownian motion, does this not contradict what we said about uniqueness of Brownian motion?

Problem 17. Let $X(t) = W(t) - tW(1)$ for $0 \le t \le 1$, where W is a Brownian motion. What are the finite dimensional distributions of X?

Problem 18. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous unbounded function with a right derivative at 0. Let $X(t) = \frac{W(\varphi(t))}{\sqrt{\varphi(t)/t}}$, a Gaussian process with zero means and $\text{Var}(X_t) = t$. Show that X is a Brownian motion if and only if $\varphi(t) = \alpha t$ for some $\alpha > 0$.

Problem 19. If W is SBM in d dimensions, for which $v \in \mathbb{R}^d$ is $t \mapsto \langle W(t), v \rangle$ a SBM in one dimension?

Problem 20. If W, B are standard Brownian motions on time interval $[0,1]$, show that the following are also standard Brownian motions on $[0,1]$. (1) $-W(t)$, (2) $W(1-t) - W(1)$, (3) $(W(t) + B(t))/\sqrt{2}$, (4) $W(t)\mathbf{1}_{t\le\frac{1}{2}} + (W(1/2) - B(1/2) + B(t))\mathbf{1}_{t>\frac{1}{2}}$, (5) $W(t)\mathbf{1}_{t\le\frac{1}{2}} + (W(1/2) + B(t-1/2))\mathbf{1}_{t>\frac{1}{2}}$.

Problem 21. Let $W = (W_t)_{0\le t\le 1}$ be a standard Brownian motion and let $0 < s < 1$ be fixed. Let $X_t = W_{t+s} - W_s$ for $0 \le t \le 1-s$ and $X_t = W_1 - W_s + W_{t-(1-s)}$ for $1-s \le t \le 1$. Show that X is a standard Brownian motion.

Problem 22. Let W_0 be standard Brownian bridge. Fix $s \in (0,1)$ and define $X(t) = W_0(t+s) - W_0(s)$ for $0 \le t \le 1-s$ and $X(t) = W_0(t-(1-s)) - W_0(s)$ for $1-s \le t \le 1$. Show that X is also a standard Brownian bridge.

Problem 23. Let W be standard Brownian motion in one dimension. Show that the distributions of following pairs of $C[0,1]$-valued random variables are singular.

(1) $(W_t)_{0\le t\le 1}$ and $(2W_t)_{0\le t\le 1}$.

(2) $(W_t)_{0\le t\le 1}$ and $(W_{2t})_{0\le t\le 1}$.

(3) $(W_t)_{0\le t\le 1}$ and $(W_t + t^{1/4})_{0\le t\le 1}$.

Problem 24. The fractional Brownian motion with index $H \in (0,1)$ is a stochastic process $X = (X_t)_{0\le t\le 1}$ such that the finite dimensional distributions are jointly Gaussian with zero means and covariances given by

$$\mathbf{E}[X_t X_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Show that such a process with continuous sample paths exists and the sample paths are Hölder(γ) for every $\gamma < H$. [*Note:* Assume that the given covariance is positive semi-definite. Showing that is not straightforward.]

Problem 25. Use Kolmogorov-Centsov theorem directly to show that w.p.1., the Brownian bridge and the Ornstein-Uhlenbeck processes are Hölder($\frac{1}{2} + \varepsilon$) for any $\varepsilon > 0$ (restrict the time interval to $[0,1]$).

Problem 26. Let $Q_n(t) := \sum_{k=0}^{\lfloor t2^n \rfloor} (W((k+1)/2^n) - W(k/2^n))^2$ where W is standard BM. For fixed $t > 0$, show that $Q_n(t) \xrightarrow{P} t$.

Problem 27. Let $Q_n(t)$ be as in the previous problem. Show that $(Q_n(t))_n$ is a reverse martingale and hence deduce that $Q_n(t) \stackrel{a.s.}{\to} t$.

Problem 28. Let W be standard Brownian motion. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be a compactly supported continuous function (this is more than needed). Show that $X_f = \int_0^\infty f(t)W(t)dt$ is a Gaussian random variable and find its mean and variance. What is the covariance between X_f and X_g?

Problem 29. Show that Wiener measure is tight. In other words, given $\varepsilon > 0$, there is a compact set $K_\varepsilon \subseteq C[0,1]$ such that $\mathbf{P}\{W \in K_\varepsilon\} \geq 1 - \varepsilon$.

Problem 30. Show that any open subset of $C_0[0,1] = \{f \in C[0,1] : f(0) = 0\}$ has positive Wiener measure. In other words, show that $\mathbf{P}\{\|W - f\|_{\sup[0,1]} < \varepsilon\} > 0$ for any $f \in C_0[0,1]$ and any $\varepsilon > 0$.

Problem 31. Let $f \in C[0,1]$ and $W = (W_t)_{t \leq 1}$ be SBM in one dimension. Show that $W + f$ is nowhere differentiable w.p.1.

Problem 32. Let W be Brownian motion and let W_0 be Brownian bridge, both on $[0,1]$. If $T < 1$, show that the distributions of $(W(t))_{0 \leq t \leq T}$ and $(W_0(t))_{0 \leq t \leq T}$ are mutually absolutely continuous.

Problem 33. Let W_n be i.i.d. SBMs on $[0,1]$. Let $\widetilde{W}_n = W_n + f_n$ where $f_n \in C[0,1]$ with $f_n(0) = 0$. Let μ and $\widetilde{\mu}$ (probability measures on $C[0,1]^{\mathbb{N}}$) be the distributions of $(W_1, W_2, \ldots)$ and $(\widetilde{W}_1, \widetilde{W}_2, \ldots)$ respectively . Find conditions on $(f_n)_n$ under which $\widetilde{\mu}$ is absolutely continuous or singular to μ.

Problem 34. Let $W = (W(t))_{t \geq 0}$ and $f \in C[0,\infty)$ with $f(0) = 0$. Show that the distribution of $W + f$ is absolutely continuous to the distribution of W if $f(t) = \int_0^t g(s)ds$ for some $g \in L^2(\mathbb{R}_+)$ and singular otherwise. [*Hint:* You may use the previous problem]

Problem 35. If μ and ν are two probability measures on $(\Omega, \mathcal{F})$ and $d\nu = g d\mu$, the *relative entropy* (also called Kullback-Leibler divergence) is defined as $D(\nu\|\mu) = \int_\Omega g \log g \, d\nu$.

(1) Find $D(\mu_b\|\mu_0)$ if $\mu_b = N_d(b, I_d)$ for $b \in \mathbb{R}^d$.

(2) Find $D(\mu_f\|\mu_f)$ if μ_f is the distribution of $W + f$, where $W = (W_t)_{0 \leq t \leq 1}$ is SBM and f is in the Cameron-Martin space.

Problem 36. Let W_0 be the Brownian bridge. For $f \in C[0,1]$ with $f(0) = 0 = f(1)$, let μ_f denote the distribution of $W_0 + f$. For which f are μ_f and μ_0 singular and for which f are the mutually absolutely continuous? [*Note:* Go over the proof of Cameron-Martin theorem and make appropriate modifications]

Problem 37. Let $f : [0, \infty) \to \mathbb{R}$ be a C^1 function. Show that

$$\int_0^t f(s)dW(s) = f(t)W(t) - \int_0^t W(s)f'(s)ds.$$

[*Remark:* Our definition of the Wiener integral on the left side was via approximation by step functions.]

Problem 38. Let $f \in \bigcap_{T<\infty} L^2[0, T]$. Let $M_t = \int_0^t f(s)dW(s)$ and let $A_t = \int_0^t f^2(s)ds$. Show that M_t and $M_t^2 - A_t$ are both martingales.

Problem 39. Define the process $X = (X(t))_{t \geq 0}$ by $X(t) = \int_0^\infty (s^\alpha - (s-t)_+^\alpha)dW(s)$.

(1) Show that this is a well-defined process for $-\frac{1}{2} < \alpha < \frac{1}{2}$.

(2) Show that $\mathbf{E}[|X(t) - X(s)|^2] = c_\alpha |t - s|^\alpha$ for some constant $c_\alpha > 0$.

(3) Deduce that X can be modified to become the fractional Brownian motion.

Problem 40. Recall that $M_t = \max\{W_s : s \leq t\}$. Let $T = \min\{t \geq 0 : W_t = M_1\}$. Show that $W(T + t) - W(T)$ is *not* a Brownian motion.

Problem 41. Let $\tau = \max\{t \leq 1 : W_t = 0\}$. Show that $W(T + t) - W(T)$ is *not* a Brownian motion.

Problem 42. Let $M = \max\{W_t : 0 \leq t \leq 1\}$. Show that M has a continuous distribution.

Problem 43. Let $\tau = \min\{t : W_t = 1\}$. Show that $\tau < \infty$ a.s.

Problem 44. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function and let $V_f = \limsup_{t \downarrow 0} \frac{W(t)}{f(t)}$.

(1) Show that V_f is a constant, a.s.

(2) Give a non-trivial f (strictly increasing function such that $f(0) = 0$) for which $V_f = 0$ a.s. and another for which $V_f = \infty$ a.s.

Problem 45. Let W be 2-dimensional SBM. Show that the event that $W[0,\varepsilon] \subseteq H$ for some $\varepsilon > 0$ and some half-space H has zero probability. [*Note:* A half-space is a set of the form $H = \{x \in \mathbb{R}^2 : \langle x, u \rangle \geq 0\}$ for some unit vector $u \in \mathbb{R}^d$.]

Problem 46. Let W be d-dimensional SBM. Show that $W[0,\varepsilon] \cap C \neq$ for any open cone C and any $\varepsilon > 0$, almost surely. By an open cone, we mean a set of the form $\{x \in \mathbb{R}^d : \langle x, v \rangle < r\}$ for some unit vector v and some $r > 0$ (in particular, the origin is not in the cone).

Problem 47. Let W be SBM. Show that almost surely, there exist (random) times $t_1 > t_2 > \ldots \to 0$ such that $W(t_{2j}) > t_{2j}^{1/4}$ and $W(t_{2j-1}) < t_{2j-1}^{1/4}$ for all j.

Problem 48. Let $W_1, \ldots, W_n$ be independent Brownian motions started at $x_1 < \ldots < x_n$. Let E be the event that $W_i(t) \neq W_j(t)$ for all $i \neq j$ and for all $t \leq T$. Show that the joint density of $(W_1(T), \ldots, W_n(T))$ on the event E is equal to $\det(p_T(y_j - x_i))_{i,j \leq n}$ at $(y_1, \ldots, y_n)$ with $y_1 < \ldots < y_n$. Here $p_T(z) = \frac{1}{\sqrt{2\pi T}} e^{-z^2/2T}$.

Problem 49. Does Blumenthal's zero-one law hold for any RCLL process?

Problem 50. Let W be one-dimensional SBM and let $Z_x = \{t : W(t) = x\}$ for any $x \in \mathbb{R}$.

(1) Show that Z_x is unbounded, a.s.

(2) Show that every point of Z_x is an accumulation point.

Problem 51. Let R be the first zero of SBM after time $t = 1$. What is the density of R? [*Hint:* Use the arcsine law for the last zero on $[0, 1]$]

Problem 52. Let $W_t = (X_t, Y_t)$ be SBM in $\mathbb{R}^2$. Let $\tau = \inf\{t : Y_t = -1\}$. Show that X_τ has Cauchy distribution with density $\frac{b}{\pi(b^2+x^2)}$ for some $b > 0$. [*Note:* In fact $b = 1$. Extra points if you can show that!]

Problem 53. We know that W_t and $t((W_t/\sqrt{t})^2 - 1)$ are martingales. Find similar martingales that are of the form $t^{d/2} Q_d(W_t/\sqrt{t})$ where Q_d is a polynomial of degree d, for $d = 3, 4$.

Problem 54. Let $\tau = \inf\{t \geq 0 : W(t) = 1 + 2t\}$. Show that $\mathbf{P}\{\tau < \infty\} < 1$.

Problem 55. Let B be planar Brownian motion started at $(1,1)$ and let τ be the first time that B hits one of the co-ordinate axes. Show that $\mathbf{E}[\tau^p] < \infty$ if and only if $p < 1$.

Problem 56. Let $0 < a < x < b$. Find the probability that Brownian motion started at x has probability hits the parabola $\sqrt{a+t}$ before hitting the parabola $\sqrt{b+t}$, and vice versa.

Problem 57. Let W_0 be standard Brownian bridge and let $T = \arg\max_{0\le t\le 1} |W_0(t)|$. Show that $T \sim \text{Unif}[0,1]$.

Problem 58. Let $A \subseteq \mathbb{R}^d$ be a set of zero Lebesgue measure. Let W be SBM in d-dimensions. Show that $\{t \ge 0 : W(t) \in A\}$ has zero Lebesgue measure (in $\mathbb{R}_+$), almost surely.

Problem 59. Let X_t be a continuous, square integrable submartingale and assume (the fact) that we can write $X_t = M_t + A_t$ where M is a continuous martingale and A_t is an increasing continuous process.

(1) Compute $\mathbf{E}[X_t \mid \mathcal{F}_s]$ and derive the formal relation $\frac{d}{du}A_{t+u}\big|_{u=0} = \frac{d}{du}\mathbf{E}[X_{t+u} \mid \mathcal{F}_t]\big|_{u=0}$.

(2) Apply this to $X_t = (W_t^2 - t)^2$ to show that $(W_t^2 - t)^2 - \int_0^t 4W_s^2 ds$ is a martingale.

Problem 60. Let W^0 be standard Brownian bridge. Show that $\left(\frac{W^0(t)}{1-t}\right)_{0\le t<1}$ is a martingale.

Problem 61. Let $M_t = \max_{0\le s\le t} W_s$ be the running maximum of Brownian motion.

(1) Show that $\mathbf{P}\{M_T \ge b\sqrt{2T\log\log T}\} \le \frac{C}{(\log T)^b}$ for large enough T.

(2) Let $b > \lambda > 1$ and let $T_k = \lambda^k$. Show that $M_{T_{k+1}} < b\sqrt{2T_k \log\log T_k}$ for all but finitely many k, almost surely.

(3) Conclude that $\limsup_{t\to\infty} \frac{W_t}{\sqrt{2\log\log t}} \le 1$ a.s.

Problem 62. Let Z_n be i.i.d. $N(0,1)$ variables and form the series $F_N(t) := \sqrt{2}\sum_{n=1}^N Z_n \frac{\sin \pi n t}{n}$ for $N \ge 1$ and $t \in [0,1]$.

(1) Show that $|F_N(t) - F_M(t)|^2 \le \sum_{j=M+1}^N \frac{Z_j^2}{j^2} + \sum_{\ell=1}^{N-M-1}\sum_{j=M+1}^{N-\ell} \left|\frac{Z_j Z_{j+\ell}}{j(j+\ell)}\right|$.

(2) Deduce that $\mathbf{E}[\|F_{2N} - F_N\|_{\sup}^2] \le \frac{C}{N^q}$ for some $q > 0$ and $C < \infty$.

(3) Conclude that F_N converges uniformly on $[0,1]$, a.s. and hence defines a random continuous function F_∞.

(4) Use the identity $\sum_{n\ge 1} \frac{\sin(\pi n n t)\sin(\pi n s)}{n^2} =$ complete to say that F_∞ is a Brownian bridge.

Problem 63. Let μ be a finite measure on $[0,1]$ and let $L(f) = \int f d\mu$.

(1) Show that L is continuous on $C[0,1]$.

(2) Conclude that $\int \mathcal{S}_n(t) d\mu(t) \xrightarrow{d} \int W(t) d\mu(t)$. Here $\mathcal{S}_n$ is the continuous piecewise linear function that takes the values $S_k/\sqrt{n}$ at times k/n, $0 \le k \le n$, where $S_k = X_1 + \ldots + X_k$ with X_k i.i.d. with zero mean and unit variance.

Problem 64. Let X_k be i.i.d. with zero mean and unit variance and let $\mathcal{S}_n(t) = \frac{S_k}{\sqrt{n}}$ for $t = \frac{k}{n}$ (for $0 \le k \le n$) and let $\mathcal{S}_n$ be linear in each interval $[(k-1)/n, k/n]$ so that $\mathcal{S}_n$ is continuous.

(1) If $\varphi \in C[0,1]$, show that $\int_0^1 \varphi(t)\mathcal{S}_n(t)dt \xrightarrow{d} \int_0^1 \varphi(t)W(t)dt$.

(2) Use the first part to show that $\frac{1}{n^{3/2}} \sum\limits_{k=1}^{n} kX_k \xrightarrow{d} N(0, \frac{1}{3})$.

Problem 65. Can you generalize the strategy in the previous problem to deduce that

$$\frac{1}{n^{(p+1)/2}} \sum_{k=1}^{n} k^p X_k \xrightarrow{d} N(0, \sigma_p^2).$$

Problem 66. Let X_k be i.i.d. random variables. If $g \in C^1[0,1]$, show that $\frac{1}{n^p} \sum_{k=1}^{n} X_k g(k/n) \xrightarrow{d} N(0, \sigma^2)$ for appropriate choice of p and σ^2 (which you must find).

Problem 67. Let U_n be i.i.d. Uniform$[0,1]$ and let $F_n(t) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{U_k \le t}$. It is a fact that the RCLL processes $\sqrt{n}(F_n(t) - t)_{0 \le t \le 1}$ converge in distribution to a $C[0,1]$-valued random variable. Find the limit.

Problem 68. Let $\mu = 0.4\,\delta_1 + 0.3\,\delta_{-2} + 0.1\,\delta_{-3} + 0.2\,\delta_b$ (point masses $0.4, 0.3, 0.1, 0.2$ at $1, -2, -3, b$ respectively).

(1) For what value of b does there exist a stopping time τ with finite mean such that $W(\tau) \sim \mu$ and what is $\mathbf{E}[\tau]$ in that case?

(2) For the above choice of b, describe as explicitly as possible a stopping time τ that does not use any extra randomness other than W.

Problem 69. Find stopping time τ(without randomization) so that $W(\tau) \sim \frac{1}{2}N(-2,1) + \frac{1}{2}N(2,1)$.

Problem 70. Let $a < 0 < b$ and let τ be the first time that a standard Brownian motion hits a or b. Find $\mathbf{E}[\tau]$ and $\mathbf{E}[\tau^2]$. [*Note:* Do it for $b = -a$ if not in general]

Problem 71. Prove Skorokhod embedding for finitely supported distributions by completing the following induction steps[1]. Let $\mu = p_1\delta_{a_1} + \ldots + p_k\delta_{a_k}$ be a probability measure with zero mean.

(1) For $k = 2$ we may take $\tau = \tau_{a_1} \wedge \tau_{a_2}$ (the standard gambler's ruin problem).

(2) If $k \geq 3$, then find b such that $\mathbf{P}\{\tau_{a_1} < \tau_b\} = p_1$. Set $\tau = \tau_{a_1}$ on the event $\tau_{a_1} < \tau_b$. On the complementary event, let $\tau = \tau_b + \tau'$, where τ' solves the embedding problem for the reduced measure (supported on $k-1$ points)
$$\nu = \frac{1}{1-p_1}(p_2\delta_{a_2-b} + \ldots + p_k\delta_{a_k-b})$$
w.r.t. the Brownian motion $W(\cdot + \tau_b) - W(\tau_b)$.

(3) From the explicit form of b and $\mathbf{E}[\tau'] = \int x^2 d\nu(x)$, check that $\mathbf{E}[\tau] = \int x^2 d\mu(x)$.

Problem 72. Let $\mathcal{L}(A) = \lambda\{t \leq 1 : W(t) \in A\}$ for $A \in \mathcal{B}(\mathbb{R})$.

(1) Show that $\mathcal{L}(\cdot)$ and $\mathbf{E}[\mathcal{L}(\cdot)]$ are measures.

(2) Find an expression for $\mathbf{E}[\mathcal{L}(A)]$.

Problem 73. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function. Let $V_\varphi = \lambda\{t \geq 0 : W(t) \geq \varphi(t)\}$.

(1) Give a reasonable sufficient condition on φ that ensures that $V_\varphi < \infty$ a.s.

(2) For $\varphi(t) = t$, find $\mathbf{E}[V_\varphi]$ and $\mathrm{Var}(V_\varphi)$.

Problem 74. Let $X_{n,j}$, $1 \leq j \leq n < \infty$, be a triangular array of random variables satisfying the conditions of Lindeberg-Feller central limit theorem.

Problem 75. Show that $\frac{1}{\|x\|}$ is harmonic on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ but $\left(\frac{1}{\|B_t\|}\right)_{t\geq 1}$ is not a martingale for 3-dimensional standard Brownian motion. [*Note:* To contrast with the point that if u is harmonic on all of $\mathbb{R}^3$ and does not grow too fast, then $u(B_t)$ is a martingale.]

[1] Aditya Priya gave this beautiful method in a solution to a problem in the 2023 final exam.