PROBLEM SET: MARTINGALES

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Note: Unless otherwise stated, the underlying probability space is denoted $(\Omega, \mathcal{F}, \mathbf{P})$. Letters $\mathcal{G}, \mathcal{H}, \ldots$ denote sub sigma algebras and $X, Y, \ldots$ denote random variables.

Problem 1. Let X have density f on $\mathbb{R}$. Find $\mathbf{E}[X \mid |X|]$.

Problem 2. Let $\mathcal{G} = \sigma(Y)$ where Y is a $\mathbb{R}^d$-valued random vector. Show that $\mathbf{E}[X \mid Y]$ is any random variable of the form $f(Y)$ where $f : \mathbb{R}^d \to \mathbb{R}$ is a Borel measurable function such that $\mathbf{E}[Xh(Y)] = \mathbf{E}[f(Y)h(Y)]$ for all bounded measurable $h : \mathbb{R}^d \to \mathbb{R}$.

Problem 3. Let $X_1, \ldots, X_n$ be i.i.d. Uniform$[0,1]$ random variables. Let R_k be the rank of X_k (i.e., $R_k = 1$ if X_k is the largest, and $R_k = n$ if X_k is the smallest). Find $\mathbf{E}[X_1 \mid R_1, \ldots, R_n]$.

Problem 4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with finite mean. Let $S_k = X_1 + \ldots + X_k$. Find $\mathbf{E}[X_1 \mid S_n]$.

Problem 5. Let X, Y be i.i.d. Exp(1) random variables. If $S = X + Y$, show that $\mathbf{E}[\varphi(X) \mid S] = \frac{1}{S}\int_0^S \varphi(t)dt$ for any bounded measurable $\varphi : \mathbb{R} \to \mathbb{R}$.

Problem 6. Let $X \sim$ Exp(λ). Let $Y_t = \mathbf{1}_{X \geq t}$ for $t \geq 0$. Show that $\mathbf{E}[X \mid \sigma(Y_t)] = Y_t + \frac{1}{\lambda}$.

Problem 7. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are conditionally independent given $\mathcal{G}_0$ if $\mathbf{P}\{A \cap B \mid \mathcal{G}_0\} = \mathbf{P}\{A \mid \mathcal{G}_0\}\mathbf{P}\{B \mid \mathcal{G}_0\}$ a.s. for any $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. Show by examples that independence and conditional independence do not imply each other.

Problem 8. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub sigma-algebra. If $A \in \mathcal{F}$ and $\mathbf{E}[\mathbf{1}_A \mid \mathcal{G}] > 0$, then there is a probability measure Q on $\mathcal{F}$ for which A is independent of $\mathcal{G}$.

Is the analogous statement true if $X > 0$ and $\mathbf{E}[X \mid \mathcal{G}] > 0$.

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Problem 9. Suppose X, Y, Z are independent random variables and $U = f(X, Z)$ and $V = g(Y, Z)$ for some bounded measurable functions f, g. Show that $\mathbf{E}[UV \mid Y] = \mathbf{E}[U \mid Y]\mathbf{E}[V \mid Y]$ (i.e., U, V are conditionally independent given Y.

Problem 10. Let $\mathcal{G}_1, \mathcal{G}_2$ be sub-sigma algebras of $\mathcal{F}$. Show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent if and only if $\mathbf{E}[X \mid \mathcal{G}_1] = \mathbf{E}[X]$ for all $\mathcal{G}_2$ measurable, integrable random variables X.

Problem 11. Let X, Y be integrable random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-sigma algebra and $A \in \mathcal{G}$. If $X = Y$ on A, show that $\mathbf{E}[X \mid \mathcal{G}] = \mathbf{E}[Y \mid \mathcal{G}]$ *a.e.*[$\mathbf{P}$] on A. [*Remark:* This is called locality property of conditional expectation.]

Problem 12. If X, Y are square integrable and $\mathbf{E}[X \mid Y] = Y$ and $\mathbf{E}[Y \mid X] = X$, then show that $X = Y$ a.s.

[*Harder:* Show the same assuming only that X, Y are integrable]

Problem 13. Let X, Y be real-valued random variables such that Y is $\mathcal{G}$-measurable and X is independent of Y. If $X \sim \mu$, then show that for any bounded measurable $F : \mathbb{R}^2 \to \mathbb{R}$, we have $\mathbf{E}[F(X, Y) \mid \mathcal{G}] = \int_{\mathbb{R}} F(x, Y) d\mu(x)$.

Problem 14. If X is square integrable and $\mathcal{G}_n' \subseteq \mathcal{G}_n \subseteq \mathcal{G}_n''$. If $\mathbf{E}[X \mid \mathcal{G}_n'] \overset{a.s.}{\to} Y$ and $\mathbf{E}[X \mid \mathcal{G}_n''] \overset{a.s.}{\to} Y$ for some Y, then show that $\mathbf{E}[X \mid \mathcal{G}_n] \overset{a.s.}{\to} Y$.

[*Harder:* Show the same assuming only that X is integrable]

Problem 15. Prove the following statements. The setting is $(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-sigma algebra.

(1) Conditional Markov inequality: If $X \geq 0$ and integrable, then $\mathbf{P}(X \geq t \mid \mathcal{G}) \leq \frac{1}{t}\mathbf{E}[X \mid \mathcal{G}]$ *a.s.*[$\mathbf{P}$].

(2) Conditional Cauchy-Schwarz inequality: If X, Y are square integrable, then $\mathbf{E}[XY \mid \mathcal{G}]^2 \leq \mathbf{E}[X^2 \mid \mathcal{G}]\mathbf{E}[^2Y \mid \mathcal{G}]$ *a.s.*[$\mathbf{P}$].

(3) Analysis of variance: If X has finite variance, then $\text{Var}(X) = \mathbf{E}[\text{Var}(X \mid \mathcal{G})] + \text{Var}(\mathbf{E}[X \mid \mathcal{G}])$.

Problem 16. State appropriate assumptions on a random variable X so that $\mathbf{E}[\log X \mid \mathcal{G}] \leq \log \mathbf{E}[X \mid \mathcal{G}]$ (the key point is to state all the assumptions you need).

Problem 17. Let (X,Y) have density $f(x,y)$ on $\mathbb{R}^2$. Find the conditional distribution of X given Y. Find the conditional expectation of $\varphi(X)$ given Y for any bounded measurable $\varphi : \mathbb{R} \mapsto \mathbb{R}$.

Problem 18. Let $X \sim N_n(\mu, \Sigma)$. Write $X = (X_1, \ldots, X_n)^t$. For $m < n$, let $Y = (X_1, \ldots, X_m)^t$ and $Z = (X_{m+1}, \ldots, X_n)^t$. Find the conditional distribution of Z given Y as explicitly as possible. In particular, work out the case when $m = n-1$.

Problem 19. Let $X_1, \ldots, X_n$ be i.i.d. Ber(p) random variables. Find the conditional distribution of $(X_1, \ldots, X_n)$ given $S := X_1 + \ldots + X_n$.

Problem 20. Let λ be the Lebesgue measure on $([0,1], \mathcal{B})$. Let $E \subseteq [0,1]$ be a non-measurable set such that the outer measure $\lambda^*(E) = 1$ and $\lambda^*(E^c) = 1$. Let $\mathcal{F} = \sigma(\mathcal{B} \cup \{E\}) = \{(A \cap E) \sqcup (B \cap E^c) : A, B \in \mathcal{B}\}$.

(1) Show that $\mu((A \cap E) \sqcup (B \cap E^c)) = \frac{1}{2}\lambda(A) + \frac{1}{2}\lambda(B)$ defines a probability measure on $\mathcal{F}$.

(2) Show that regular conditional probability $\mu(\cdot \mid \mathcal{B})$ does not exist.

Problem 21. (1) Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean. Let $S_n = X_1 + \ldots + X_n$. Show that $\mathbf{E}[X_1 \mid S_n] = \frac{1}{n} S_n$.

(2) If $X \sim \text{Exp}(\lambda)$, and $Y = \mathbf{1}_{X \geq t}$, find the conditional distribution of X given Y.

(3) If X, Y are i.i.d. Exp(λ), find the conditional distribution of $\frac{X}{X+Y}$ given $X+Y$.

Problem 22. Let $(\Omega_i, \mathcal{F}_i)$ be two measure spaces and let ν be a probability measure on Ω_1 and let $\kappa : \Omega_1 \times \mathcal{F}_2 \mapsto [0,1]$ be a stochastic kernel (as defined in class). Define a probability measure μ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ by $\mu(A \times B) = \int_A \left[\int_B \kappa(x, dy)\right] \nu(dx)$ for cylinder sets $A \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. Let X, Y be the projections from $\Omega_1 \times \Omega_2$ to Ω_1 and to Ω_2, respectively.

(1) Find the conditional distribution of Y given X.

(2) Find the conditional distribution of X given Y. (This is called *Bayes' rule*).

Problem 23. Let X be an integrable random variable on $(\Omega, \mathcal{F}, \mathbf{P})$. Suppose $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ are sub-sigma algebras.

(1) If $\mathbf{E}[X \mid \mathcal{G}_1]$ has the same distribution as $\mathbf{E}[X \mid \mathcal{G}_2]$, then $\mathbf{E}[X \mid \mathcal{G}_1] = \mathbf{E}[X \mid \mathcal{G}_2]$ *a.s.*

(2) Assume X has finite variance. If $\mathbf{E}[X \mid \mathcal{G}_1]$ has the same variance as $\mathbf{E}[X \mid \mathcal{G}_2]$, then $\mathbf{E}[X \mid \mathcal{G}_1] = \mathbf{E}[X \mid \mathcal{G}_2]$ *a.s.*

Problem 24. Let $X_1, \ldots, X_n$ be i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{S} = \sigma\{f(X_1, \ldots, X_n) : f : \mathbb{R}^n \mapsto \mathbb{R}$ is measurable and symmetric$\}$. Find the conditional expectation and conditional distribution of X_1 given $\mathcal{S}$.

What if we condition on $\mathcal{A}$, the sigma-algebra generated by all *anti-symmetric* functions of $X_1, \ldots, X_n$ or on $\mathcal{C}$, the set of cyclically symmetric functions of $X_1, \ldots, X_n$? Recall that an anti-symmetric function changes sign if two of the arguments are interchanges. A cyclically symmetric function is invariant under cyclic permutations of the co-ordinates.

Problem 25. Let X, Y, Z be i.i.d. real-valued random variables with density f. Let $m = \min\{X, Y, Z\}$ and $M = \max\{X, Y, Z\}$ and let $\mathcal{G} = \sigma\{m, M\}$. Find $\mathbf{E}[X \mid \mathcal{G}]$.

Problem 26. Let X, Y be i.i.d. random variables with distribution μ and let $\mathcal{G} = \sigma\{X + Y\}$. Find the conditional distribution of X when μ is (a) $N(0, 1)$, (b) $\text{Exp}(\lambda)$, (c) $\text{Poisson}(\lambda)$.

Problem 27. Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean and let $S_k = X_1 + \ldots + X_k$ (and $S_0 = 0$). Let $\mathcal{G}_n = \sigma\{S_0, \ldots, S_n\}$ and $\mathcal{H}_n = \sigma\{S_n, S_{n+1}, \ldots\}$. Find (1) $\mathbf{E}[S_n \mid \mathcal{G}_m]$, (2) $\mathbf{E}[S_n \mid \mathcal{H}_m]$, (3) $\mathbf{E}[X_n \mid \mathcal{G}_m]$ and (4) $\mathbf{E}[X_n \mid \mathcal{H}_m]$.

Problem 28. Let $X = (X_n)_{n \geq 0}$ be a Markov chain on a finite state space S with transition matrix $P = (p_{i,j})_{i,j \in S}$. Let $f : S \to \mathbb{R}$ be a function. Write an expression for $\mathbf{E}[f(X_n) \mid X_0, \ldots, X_{n-1}]$ in terms of the transition matrix.

Problem 29. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathbf{Q}$ be another probability measure on $\mathcal{F}$ such that $d\mathbf{Q} = Y d\mathbf{P}$ (Y is the Radon Nikodym derivative). If X is a random variable integrable with respect to $\mathbf{Q}$, then XY is integrable with respect to $\mathbf{P}$ and for any $\mathcal{G} \subseteq \mathcal{F}$,

$$E_{\mathbf{Q}}[X \mid \mathcal{G}] = \frac{\mathbf{E}_{\mathbf{P}}[XY \mid \mathcal{G}]}{\mathbf{E}_{\mathbf{P}}[Y \mid \mathcal{G}]}.$$

Problem 30. Let S_1, S_2 be finite sets. Suppose $\mu : S_1 \mapsto \mathcal{P}(S_2)$ and $\nu : S_2 \mapsto \mathcal{P}(S_1)$ are stochastic kernels (i.e., μ_x, $x \in S_1$, are probability measures on S_2 and ν_y, $y \in S_2$, are probability measures on S_1). The question is whether there exists a probability distribution α on $S_1 \times S_2$ such that if $(X, Y) \sim \alpha$, then μ_x is the conditional distribution of Y given $X = x$ and ν_y is the conditional distribution of X given $Y = y$.

Show that the answer is yes if and only if the function $(x, y) \mapsto \frac{\mu_x\{y\}}{\nu_y\{x\}}$ factors as a function of x times a function of y.

Problem 31. Let X_n be SRW on $\mathbb{Z}$ with probability p of going up. and probability $q = 1 - p$ of going down, at each step. Show that $(q/p)^{X_n}$ is a martingale.

Problem 32. Let ξ_n are i.i.d. random variables with zero mean and finite moments of all orders.

(1) Show that $M^{(k)}(n) := \sum_{1 \le i_1 < \ldots < i_k < n} \xi_{i_1} \ldots \xi_{i_n}$ is a martingale.

(2) Hence or otherwise, find martingales of the form $P_{n,k}(S_n)$ where $P_{n,k}$ is a polynomial of degree k, for $k = 3, 4$ (we have seen in class that we may take $P_{n,1}(S_n) = S_n$ and $P_{n,2}(S_n) = S_n^2 - n$).

Problem 33. Consider a finite graph $G = (V, E)$ in which each vertex has degree d. Let U_n be i.i.d. unif$[0, 1]$ random variables. Setting S_0 to be any subset of V, successively define S_{n+1} to be be the subset of all vertices that have at least dU_n neighbours in S_n (this means that if $S_n = \emptyset$ or V, then $S_{n+1} = \emptyset$ or V, respectively). Show that $|S_n|$ (the cardinality of S_n) is a martingale.

Problem 34. If $X = (X_n)_{n \ge 0}$ is a submartingale, then is it necessarily true that $(X_n - \mathbf{E}[X_n])_{n \ge 0}$ is a martingale?

Problem 35. Suppose $X = (X_n)$ and $Y = (Y_n)$ are submartingales (same filtration). Which of the following are necessarily sub-martingales? (1) $X_n + Y_n$, (2) $X_n - Y_n$, (3) $X_n Y_n$, (4) $X_n \wedge Y_n$, (5) $X_n \vee Y_n$?

Problem 36. *Doob decomposition:* Let X be a sub-martingale. Show that there is a unique pair of processes (M, A) such that M is a martingale, A is a predictable increasing process with $A_0 = 0$ and such that $X_n = M_n + A_n$. [*Remark:* Here the filtration is fixed throughout. Also, the uniqueness is up to sets of zero probability (make that precise).]

Problem 37. Let $(M_n)_{n \ge 0}$ be a martingale and let H be a predictable process. We know that $X_n = \sum_{k=1}^{n} H_k (M_k - M_{k-1})$ is a martingale, and hence X_n^2 is a sub-martingale.

(1) Find the Doob decomposition of X_n^2.

(2) Special to the case when $M_n = \xi_1 + \ldots + \xi_n$, where $\xi_k \sim N(0, 1)$ are i.i.d.

Problem 38. Let X be a submartingale w.r.t. $\mathcal{F}_\bullet = (\mathcal{F}_n)_{n \ge 0}$. Suppose $\mathcal{G}_\bullet = (\mathcal{G}_n)_{n \ge 0}$ is another filtration such that (a) $\mathcal{G}_n \supseteq \mathcal{F}_n$ for each n and (b) If $A \in \mathcal{G}_n$, there is some $B \in \mathcal{F}_n$ such that $\mathbf{P}(A \Delta B) = 0$.

Show that X is a submartingale w.r.t. $\mathcal{G}_\bullet$.

Problem 39. Let ξ_n be i.i.d. Ber(p) and let $Y_0 > 0$ and $Y_n = Y_{n-1} + H_n \xi_n$ where H is a predictable process (for the natural filtration of ξ_ns). Impose the constraint $0 \le H_n \le Y_{n-1} \vee 0$ (if Y_{n-1} is the fortune after $n-1$ games, cannot bet more than that one has on the nth game). Define $r_n = \frac{1}{n}\mathbf{E}[\log(Y_n/Y_0)]$ as the rate of increase of the gambler's fortune per game, in the first n games.

(1) Show that $\log Y_n - nb$ is a supermartingale, where $b = p\log p + (1-p)\log(1-p) + 2\log 2$. Hence conclude that $r_n \le b$.

(2) Find H_n so that $r_n = b$ for all n. This is the optimal strategy.

Problem 40. Consider $([0,1), \mathcal{B}, \lambda)$ and let $\mathcal{F}_n$ be the sigma algebra generated by the partition $[(j/2^n, (j+1)/2^n), 0 \le j < 2^n - 1$, for $n \ge 1$. Let $X_k(t)$ denote the kth digit in the binary expansion of t (for dyadic rationals, we take the terminating expansion). Let $\mathcal{G}_\bullet$ be the natural filtration generated by $X = (X_k)_{k\ge 1}$.

Show that $\mathcal{G}_\bullet = \mathcal{F}_\bullet$.

Problem 41. Consider $([0,1), \mathcal{B}, \lambda)$ and let $X_k(t)$ denote the kth digit in the binary expansion of t (for dyadic rationals, we take the terminating expansion). Let $\mathcal{F}_\bullet$ be the natural filtration generated by $X_1, X_2, \ldots$. Let $\tau = \min\{k \ge 1 : X_k = 1\}$.

(1) Show that τ is a stopping time that is finite w.p.1.

(2) Can you describe $\mathcal{F}_\tau$? (for example, by giving a partition that generates it).

Problem 42. Let X be a Markov chain on state space S and let $F : \sqcup_{n\ge 0} S^{n+1} \to \mathbb{Z}$ be a function. Let $\tau = \min\{n \ge 0 : F(X_0, \ldots, X_n) = 0\}$. Show that τ is a stopping time (for the natural filtration).

Problem 43. Solve the previous problem if X_k are the digits in the base-3 expansion and $\tau = \min\{k : X_k = 2\}$.

Problem 44. Let X be a Markov chain on a state space S. Which of the following is a stopping time?

(1) $\tau_{i,j} = \min\{n : (X_n, X_{n+1}) = (i,j)\}$,

(2) $\tau'_{i,j} = \min\{n : (X_{n-1}, X_n) = (i,j)\}$.

Problem 45. Let X be simple random walk on the complete graph K_n and let $\tau_i = \min\{n : X_n = i\}$ and $\sigma_{j,k} = \min\{n : X_{n-1} = j, X_n = k\}$. Under what conditions are $\mathcal{F}_{\tau_i}$ and $\mathcal{F}_{\sigma_{j,k}}$ contained in one another? Here $\mathcal{F}_\bullet$ is the natural filtration of the Markov chain. [*Remark:* If convenient, assume that $X_0 = \ell$ that is different from i, j, k.]

Problem 46. Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbf{P})$ be a filtered probability space and let X be a sub-martingale. Let $\tau_1 \le \tau_2$ be two stopping times such that $\tau_2 \le N$ *a.s.* for some $N < \infty$.

(1) Show that $\mathbf{E}[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] \ge X_{\tau_1}$.

(2) If $\{X_{\tau_2 \wedge n}\}$ is uniformly integrable, show the same conclusion without the hypothesis that τ_2 is a bounded random variable.

Problem 47. Let X be $\mathcal{F}_\bullet$-adapted and $\mathbf{E}[|X_t|] < \infty$ for all t. Assume that $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ for all bounded stopping times τ (or even stopping times that take finitely many values). Show that X is a martingale w.r.t. $\mathcal{F}_\bullet$.

Problem 48. Let S be a random walk on $\mathbb{R}$ with i.i.d. $N(0,1)$ steps.

(1) For $\theta \in \mathbb{R}$, show that $M_n^\theta := e^{\theta S_n - \frac{1}{2}\theta^2 n}$ is a martingale.

(2) Differentiate w.r.t θ repeatedly and set $\theta = 0$ to get martingales that are polynomials in S_n and n. Evaluate the first four of these explicitly (you may show that these are martingales directly or by justifying differentiation under expectation).

Problem 49. Let $0 < \varepsilon_n < 1$ and let X_n be a random variable that takes the values $1 \pm \varepsilon_n$ with equal probability. Show that $M_n := X_1 \dots X_n$ is a martingale and find conditions on (ε_n) to ensure that M is L^p bounded for $p = 1, 2$.

Problem 50. Let L_a denote the graph with vertices $\{0, 1, \dots, a\}$ with edges between i and $i+1$ for $0 \le i \le a-1$. Fix $a_1, \dots, a_k$ and let G be the graph got by merging the 0 vertex of $L_{a_1}, \dots, L_{a_k}$ (it is a tree with one root from which paths of lengths $a_1, \dots, a_k$ emanate). Let X be SRW on G started at the root 0. Let τ be the first time that the RW hits a leaf (a leaf is a degree 1 vertex). Find the probability distribution of X_τ. [*Hint:* First solve for the right harmonic function and use that to answer the question]

Problem 51. (Gambler's ruin problem on a regular tree). Let T_n be the regular binary tree up to n generations (this is the tree where generation k has $3 \times 2^{k-1}$ individuals, for $k = 1, 2, \dots, n$, generation 0 has one individual, and every vertex has degree 3). Let B denote the vertices in the n-th generation. Solve for the harmonic measure on B from any vertex v (i.e., find $\mathbf{P}_v(X(\tau_B) = u$ for any $u \in B$).

Problem 52. Consider the random walk $S_n = X_1 + \dots + X_n$ where the steps X_k are i.i.d. with zero mean and unit variance and bounded. Let T_n be the first time that the random walk exits $[-n, n]$. Show that $\mathbf{E}[T_n] \asymp n^2$.

Problem 53. $Y_0, Y_1, \ldots$ be random variables (assume real-valued, although that is not necessary) on $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathcal{F}_n = \sigma\{Y_0, \ldots, Y_n\}$. Let τ be a $\mathcal{F}_\bullet$-stopping time. Show that $\mathcal{F}_\tau$ is the same as the sigma-algebra generated by the stopped process $\{Y_{\tau \wedge n}\}_{n \geq 0}$.

Problem 54. Go back to the problem of finding $\mathbf{E}[\tau_{101}]$ in a sequence of fair coin tosses.

(1) Write the proof given in class in mathematical terms (without reference to gamblers and betting, etc.)

(2) Find $\mathbf{E}[\tau^2]$ and $\mathbf{E}[e^{u\tau}]$ for small enough u (again, come up with an appropriate betting games).

Problem 55. Let τ_{1010} be the waiting time for the pattern 1010 to show up in a sequence of independent tossed of a p-coin. Show that $\mathbf{E}[\tau_{1010}] = \frac{1}{p^2q^2} + \frac{1}{pq}$.

Problem 56. Consider a sequence of fair coin tosses $\xi_1, \xi_2, \ldots$ and let A be a finite set of finite binary strings. Can you find $\mathbf{E}[\tau_A]$ where τ_A is the smallest n at which one of the strings in A has occurred. You may just work it out for the examples $A = \{101,\ 111\}$ or $A = \{100,\ 110\}$.

Problem 57. A fair die is rolled repeatedly. Find the expected number of rolls till the following patterns show up: (a) $(1, 2, 3, 4)$, (b) $(6, 6, 6, 6)$.

Problem 58. N people come to a party and throw their cell phones into a box. While leaving, all of them put their hands into the box and grab a random phone (each one grabs a different phone). Those who got their own phones leave with them and the rest throw back the phone they have grabbed back into the box. In the second round, the process repeats with these remaining people. And so on. Let T be the number of rounds till everyone gets his or her phone. Show that $\mathbf{E}[T] = N$.

Problem 59. Suppose r distinguishable balls are thrown independently and uniformly at random into one of m distinguishable bins. Let $E_{r,m}$ be the number of empty bins. Show that $\mathbf{P}\{|E_{r,m} - m(1 - m^{-1})^r| \geq t\sqrt{m}\} \leq e^{-ct^2}$ with an explicit constant $c > 0$.

Problem 60. Let $A \subseteq \{0, 1\}^n$ have cardinality at least $\varepsilon 2^n$. Then show that for some fixed constant B_ε, all except $\varepsilon 2^n$ points of $\{0, 1\}^n$ can be got by changing at most $B_\varepsilon \sqrt{n}$ co-ordinates of some point in A.

[*Remark:* This is called isoperimetric inequality. Although points in $\{0, 1\}^n$ can be as far as Hamming distance (i.e., ℓ^1 distance) n, what we see is that any substantial set when enlarged will almost entirely cover the whole hypercube.]

Problem 61. Alternate proof of Hoeffding's inequality.

(1) Modify the proof of the maximal inequality to show that for any $\theta > 0$,

$$\mathbf{P}\{\max_{0\le k\le n} X_k - X_0 \ge t\} \le e^{-\theta t}\mathbf{E}[e^{\theta(X_n - X_0)}].$$

(2) Use the first part to give a proof of the following stronger version of Hoeffding's inequality:

$$\mathbf{P}\{\max_{0\le k\le n} X_k - X_0 \ge t\} \le e^{-\frac{t^2}{2D^2}}$$

where $D^2 = d_1^2 + \ldots + d_n^2$ and $|X_i - X_{i-1}| \le d_i$ a.s.

Problem 62. In the following cases, determine whether $(X(\tau \wedge n))_n$ is uniformly integrable.

(1) X is SSRW on $\mathbb{Z}$ and τ is the first hitting time of 10.

(2) X is SSRW on $\mathbb{Z}$ and τ is the first hitting time of ± 10.

(3) X is SRW on $\mathbb{Z}$ with $p_{i,i+1} = \frac{3}{4}$ and $p_{i,i-1} = \frac{1}{4}$, and τ is the first hitting time of 10.

(4) Same as the previous but τ is the first hitting time of -10.

(5) X_n is the size of the nth generation of a branching process with Poisson(1) offspring distribution and τ is the first time (generation) n for which $X_n = 0$.

(6) $X_n = f(Y_n)$ where Y is a positive recurrent Markov chain and f is a bounded real-valued function on the state space, and τ is the first return time to the initial state.

Problem 63. Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, \mathbf{P})$ be a probability space. Let U be a real-valued random variable and let T be a non-negative integer valued random variable. Set $X_n = U\mathbf{1}_{T\le n}$ for $n \ge 0$.

(1) Show that X is adapted to $\mathcal{F}_\bullet$ if and only if T is a stopping time for $\mathcal{F}_\bullet$ and U is $\mathcal{F}_T$-measurable.

(2) What if we have two random variables U, V and set $X_n = U\mathbf{1}_{T\le n} + V\mathbf{1}T > n$?

Problem 64. Suppose X_n, X are integrable and $X_n \uparrow X$ *a.s.* If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ are sigma algebras and $\mathcal{F} = \sigma\{\cup_n \mathcal{F}_n\}$, then show that $\mathbf{E}[X_n \mid \mathcal{F}_n] \uparrow \mathbf{E}[X \mid \mathcal{F}]$ *a.s.*

Problem 65. Let X be a submartingale and let τ be a stopping time with finite mean. Assume that $\mathbf{E}[|X_{n+1} - X_n| \mid \mathcal{F}_n] \le M$ a.s., for some $M < \infty$ and any n. Show that $\mathbf{E}[X_T] \ge \mathbf{E}[X_0]$.

[*Note:* In class we proved this under the stronger assumption that $\{X_{k+1} - X_k\}$ is uniformly bounded.]

Problem 66. If X be an integrable random variable on $(\Omega, \mathcal{F}, \mathbf{P})$, then show that $\{\mathbf{E}[X \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F}\}$ is uniformly integrable. What about $\{\mathbf{E}[X_i \mid \mathcal{G}] : i \in I \text{ and } \mathcal{G} \subseteq \mathcal{F}\}$ where $\{X_i : i \in I\}$ is a uniformly integrable collection?

Problem 67. Let $X_n \geq 0$ be integrable random variables and let $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. Suppose $\mathbf{E}[X_n \mid \mathcal{F}_{n-1}] \leq \left(1 + \frac{1}{n^2}\right) X_{n-1}$ a.s. for each $n \geq 1$. Show that X_n converges almost surely to a finite random variable.

Problem 68. Prove Khinchine's theorem using martingales: If $\xi_1, \xi_2, \dots$ are independent random variables with zero means and $\sum_n \operatorname{Var}(\xi_n) < \infty$, then $\sum_n \xi_n$ converges almost surely.

Problem 69. Let ξ_n be i.i.d. non-negative random variables with mean 1. Assume that ξ_1 is not identically equal to 1. Let $X_n = \xi_1 \dots \xi_n$. Show that $X_n \stackrel{a.s.}{\to} 0$.

Problem 70. Let X_p be i.i.d. mean zero random variables taking values in the unit circle S^1 (e.g., ± 1 valued) indexed by prime numbers. Form the random *Euler product* $L(z) := \prod_p \frac{1}{1 - \frac{X_p}{p^s}}$. Show that L converges almost surely for fixed s with $\operatorname{Re} s > \frac{1}{2}$.

Problem 71. Let X_i be i.i.d. random variables. Then $\frac{1}{n(n-1)} \sum\limits_{1 \leq i \neq j \leq n} \sin(X_i X_j^2)$ converges almost surely to a constant.

Problem 72. Let $\theta \sim \operatorname{unif}[0,1]$ and conditional on θ, let $X_1, X_2, \dots$ be i.i.d. $\operatorname{Ber}(\theta)$. Show that $\mathbf{E}[\theta \mid X_1, \dots, X_n] \stackrel{a.s.}{\to} \theta$.

Problem 73. Prove Liouville's theorem: *A bounded harmonic function on* $\mathbb{R}^2$ *is constant*. Similarly show that a positive harmonic function on $\mathbb{R}^2$ is constant.

[*Hint:* Use the random walk on $\mathbb{R}^2$ with steps having standard Gaussian distribution on $\mathbb{R}^2$. Show that the random walk visits any open disk infinitely often, almost surely.]

Problem 74. Let T be an infinite b-regular tree where $b \geq 3$. Let $X = (X_n)_{n \geq 0}$ be SRW on T. Fix a vertex o and let its b neighbours be $u_1, \dots, u_b$. For any vertex x, define

$$f(x) = \mathbf{P}_x\{u_1 \text{ is on the path from } o \text{ to } X_n \text{ for all large enough } n\}.$$

Show that f is a non-constant bounded harmonic function. [*Note:* It is possible to find f explicitly, but this is not part of the homework exercise.]

Problem 75. Let $X_0 = 1$.

(1) For $n \geq 1$, conditional on $X_0, \dots, X_{n-1}$, suppose $X_n \sim \operatorname{Pois}(X_{n-1})$. Show that $X_n \stackrel{a.s.}{\to} 0$.

(2) For $n \geq 1$, conditional on $X_0, \ldots, X_{n-1}$, suppose $X_n \sim \text{Pois}(S_n/n)$ where $S_n = X_0 + \ldots + X_{n-1}$. Show that $X_n \xrightarrow{a.s.} 0$.

Problem 76. Let S_n be simple random walk on $\mathbb{Z}$ that moves right with probability p and left with probability $q = 1 - p$. Find and $\alpha > 0$ such that α^{S_n} is a martingale. Use it to find the probability that the random walk hits $+b$ before $-a$, (where a, b are positive integers) starting from $S_0 = 0$. Then find the probability that starting from $S_0 = 0$ the walk ever hits x, for any $x \in \mathbb{Z}$.

Problem 77. Suppose X_n is a martingale with $X_0 = 0$. Let $X_n^* = \max\{X_0, X_1, \ldots, X_n\}$. Show that $\mathbf{E}[X_n^*] \leq C_1 + C_2 \max_{k \leq n} \mathbf{E}[|X_k| \log_+ |X_k|]$.

Problem 78. Consider an urn that initially contains one red, one blue, one green ball. Following Pólya's scheme, a ball is drawn at random, and returned to the urn with another ball of the same colour. The process is repeated. Let R_n, B_n, G_n be the numbers of balls of each colour after after n steps.

(1) Show that the $\frac{1}{n}(R_n, B_n, G_n)$ converges almost surely to a random vector (r, b, g) taking values in the simplex $\Delta = \{(x_1, x_2, x_3) : x_i \geq 0 \text{ and } x_1 + x_2 + x_3 = 1\}$.

(2) Find the joint density of (r, b). [*Hint:* Show exchangeability of the colours drawn].

Problem 79. Consider a multi-coloured Pólya's urn (draw a ball and return to the urn with another ball of the same colour) with the initial urn consisting of α_k balls of colour k, where $1 \leq k \leq 5$. Show that the limiting proportions $(X_1, \ldots, X_5)$ of the colours have Dirichlet$(5; \alpha_1, \ldots, \alpha_5)$ distribution. This just means that $X_i \geq 0$, $X_1 + \ldots + X_5 = 1$ and the joint density of $(X_1, \ldots, X_4)$ is given by

$$f(x_1, \ldots, x_4) = x_1^{\alpha_1 - 1} \ldots x_4^{\alpha_4 - 1}(1 - x_1 - \ldots - x_4)^{\alpha_5 - 1} \text{ for } x_i \geq 0 \text{ and } x_1 + \ldots + x_4 \leq 1.$$

Problem 80. Let Π_n be a uniformly chosen random permutation of $\{1, 2, \ldots, n\}$. Order the cycles of Π_n in increasing order of their smallest members and let the sizes be $C_{n,1}, C_{n,2}, \ldots$. Show that

$$\frac{1}{n}(C_{n,1}, C_{n,2}, \ldots) \xrightarrow{d} (V_1, \overline{V}_1 V_2, \overline{V}_1 \overline{V}_2 V_3, \ldots)$$

where V_i are i.i.d. Unif$[0,1]$ and $\overline{V} = 1 - V$.

[*Note:* The convergence in distribution here only means that $\frac{1}{n}(C_{n,1}, C_{n,2}, \ldots, C_{n,k})$ converges in distribution to $(V_1, \overline{V}_1 V_2, \overline{V}_1 \overline{V}_2 V_3, \ldots, V_k \prod_{j=1}^{k-1} \overline{V}_j)$ for every $k \geq 1$. The hint is to use the Chinese restaurant process for constructing Π_ns.]

Problem 81. Let ξ_n be i.i.d. $\text{Ber}_{\pm}(1/2)$ random variables. Let $S_n = \xi_1 + \ldots + \xi_n$ for $n \geq 1$ and $S_0 = 0$.

(1) Find a martingale of the form $X_n = S_n^4 + a(n)S_n^2 + b(n)$ where $a(n), b(n)$ are non-random. [*Hint:* If I had asked for a martingale of the form $S_n^2 + a(n)$, then what works is $a(n) = -n$].

(2) For a positive integer M, let $\tau_M = \min\{n \geq 1 : S_n = \pm M\}$. Find the first and second moments of τ_M.

Problem 82. Consider a binary tree up to n generations (figure given below for $n = 3$).

(1) Find the harmonic function that takes the value 0 at the leaves and 1 at the root.

(2) Starting from a vertex v in the tree, what is the probability that the random walk reaches the root (vertex labelled 1) before hitting one of the leaves (in the picture they are vertices labelled $8 - 15$)?

Problem 83. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values in a Hilbert space H with $\mathbf{E}\|X_1\| < \infty$. Let $M_n = \|S_n\|$ where $S_n = X_1 + \ldots + X_n$. If M a martingale or a submartingale or a supermartingale?

Problem 84. If X_n is a martingale, according to *Doob decomposition* there is a unique way of writing $X_n^2 = M_n + A_n$ where M_n is a martingale, A_n is an increasing process and $A_0 = 0$. Find the Doob decomposition for the squares of the following martingales (here $S_n = X_1 + \ldots + X_n$ where X_k are i.i.d.): (a) S_n, assuming $\mathbf{E}[X_1] = 0$. (b) $S_n^2 - n$ assuming $\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] = 1$.

Problem 85. Let $X = (X_n)_{n \geq 0}$ be a uniformly integrable martingale. Show that we can decompose it as $X_n = Y_n - Z_n$ where $Y_n \geq 0$ and $Z_n \geq 0$ a.s. and Y, Z are uniformly integrable martingales. Is the decomposition unique?

Problem 86. *Krickeberg decomposition*: Show that the decomposition in the previous problem hold for any L^1-bounded martingale X_n by completing the following steps:

(1) Show that $\mathbf{E}[X_{k+n}^+ \mid \mathcal{F}_n]$ is decreasing in k almost surely, and denote the limit (as $k \to \infty$) as Y_n.

(2) Show that Y_n and $X_n - Y_n$ are positive martingales.

Problem 87. *Riesz decomposition:* A *potential* is a positive supermartingale such that $\mathbf{E}[Z_n] \to 0$. Show that any positive supermartingale X_n is the sum of a martingale Y_n and a potential Z_n by completing the following steps.

(1) Show that $m \mapsto \mathbf{E}[X_{n+m} \mid \mathcal{F}_n]$ is almost surely decreasing and define Y_n to be the limit.

(2) Show that Y_n is a martingale.

(3) Show that $X_n - Y_n$ is a potential.

Problem 88. Let $G = (V, E)$ be any graph with finite vertex degrees. Let $f : V \to \mathbb{R}_+$ be a positive superharmonic function. Use the Riesz decomposition for $f(S_n)$ (where S_n is the simple random walk on G) to deduce that we can write $f = g + h$ where g is a positive harmonic function and h is a positive superharmonic function such that $h(v) \to 0$ as $v \to \infty$ (meaning that the set $\{v : h(v) > \varepsilon\}$ is finite for any $\varepsilon > 0$).

[*Remark:* Observe that there are no non-constant superharmonic functions on a recurrent graph, hence this is a non-trivial statement only for (certain) transient graphs.]

Problem 89. Among all functions from $\{1, 2 \ldots, n\}$ to itself, pick one uniformly at random and denote it F. Let $N(F)$ denote the cardinality of the range of F. Show that for some $B > 0$,

$$\mathbf{P}\{|N(F) - n(1 - e^{-1})| \geq x\sqrt{n} + B\} \leq 2e^{-\frac{x^2}{2}} \qquad \text{for any } x > 0 \text{ and large enough } n.$$

Problem 90. Let X_n be independent random variables such that $S_n = X_1 + \ldots + X_n$ converges in distribution. Show that S_n converges almost surely by completing the following steps.

(1) Show that $M_n := \frac{e^{itS_n}}{\mathbf{E}[e^{itS_n}]}$ is a martingale.

(2) Show that there exists $\varepsilon > 0$ such that $|\mathbf{E}[e^{itS_n}]| \geq \frac{1}{2}$ for all $|t| \leq \varepsilon$ and for all n [*Hint:* Use Lévy's continuity theorem].

(3) Show that M_n converges almost surely and in L^1 and hence conclude that e^{itS_n} converges almost surely for each $t \in [-\varepsilon, \varepsilon]$.

(4) Deduce that S_n converges almost surely.

[*Remark:* Don't bother that things are complex-valued. Apply relevant theorems.]

Problem 91. Let $X = (X_n)_{n \geq 0}$ be an $\mathcal{F}_\bullet$-submartingale.

(1) Show that there is a unique increasing, predictable process $A = (A_n)_{n \geq 0}$ with $A_0 = 0$ such that $X_n - A_n$ is an $\mathcal{F}_\bullet$-martingale.

(2) Show that the process A depends on the filtration $\mathcal{F}_\bullet$ by giving an example.

Problem 92. *Choquet-Deny theorem*: Let μ be a probability measure on $\mathbb{R}^d$ and let . A bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be μ-harmonic if $f(x) = \int f(x + y) d\mu(y)$ for all x. Let $S_n = X_1 + \ldots + X_n$ where $X_i \sim \mu$ are i.i.d.

(1) Show that f is μ-harmonic if and only if $f(S_n + x)$ is a martingale for any $x \in \mathbb{R}^d$.

(2) Show that $f(x + S_n)$ converges almost surely and in L^1 to some Z_x.

(3) Invoke Hewitt-Savage zero-one law to deduce that $Z_x = f(x)$ a.s.

(4) Argue that $f(\cdot + x) = f(\cdot)$ for every x in the (closed) support of μ.

Problem 93. Let θ be a $[0,1]$-valued random variable $U_1, U_2, \ldots$ be i.i.d. Unif$[0,1]$ variables independent of θ. Let $\xi_k = \mathbf{1}_{U_k \le \theta}$ and let $S_n = \xi_1 + \ldots + \xi_n$.

(1) Show that S_n/n is a martingale (note that the filtration is $\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}$).

(2) Assume that $\theta \sim$ Unif$[0,1]$ and compute $\mathbf{E}[\theta \mid \mathcal{F}_n]$.

Problem 94. Let X_n be a positive supermartingale. For $0 < a < b$, let $\sigma = \min\{k : X_k \le a\}$ and $\tau = \min\{k \ge \sigma_1 : X_k \ge b\}$. Use OST for the stopping times $\sigma \wedge N$ and $\tau \wedge N$ to deduce that $\mathbf{P}\{\tau \le N\} \le \frac{a}{b}$ for any N.

Problem 95. Assume that $X = (X_n)_{n \ge 0}$ is a process with $X_0 = 0$ such that $e^{\theta X_n - \frac{1}{2}\theta^2 n}$ is a martingale, for any $\theta \in \mathbb{R}$. Deduce that X is random walk with i.i.d. $N(0,1)$ steps.

Problem 96. Let X be a real-valued, integrable random variable. For a sequence $t_n \in \mathbb{R}$, let $\mathcal{F}_n = \sigma\{\sin(t_1 X), \sin(t_2 X), \ldots, \sin(t_n X)\}$.

(1) If $t_n = n$, under what conditions on X can we conclude that $\mathbf{E}[X \mid \mathcal{F}_n] \overset{a.s.}{\to} X$?

(2) Is there a sequence (t_n) for which $\mathbf{E}[X \mid \mathcal{F}_n] \overset{a.s.}{\to} X$ for any X?

Problem 97. Let $\theta \sim$ Unif$[0,1]$ and let $U_1, U_2, \ldots$ be i.i.d. Unif$[0,1]$ random variables independent of θ. Let $\xi_k = \mathbf{1}_{U_k \le \theta}$. Find $\mathbf{E}[\theta \mid \xi_1, \ldots, \xi_n]$.

Problem 98. If $\mu \ne \nu$ are two probability measures on some $(\Omega, \mathcal{F})$, show that the i.i.d. product measures $\mu^{\otimes \mathbb{N}}$ and $\nu^{\otimes \mathbb{N}}$ on $(\Omega^{\mathbb{N}}, \mathcal{F}^{\otimes \mathbb{N}})$ are singular.

Problem 99. Let μ_t be the Exp(t) distribution. Let $\nu = \mu_1 \otimes \mu_1 \otimes \mu_1 \otimes \ldots$ and $\theta = \mu_{t_1} \otimes \mu_{t_2} \otimes \mu_{t_3} \otimes \ldots$. If $t_k = 1 + \frac{1}{k^p}$, then find the values of p for which θ is singular to ν.

Problem 100. Let μ_t be the $N(t,1)$ distribution. Let $\nu = \mu_0 \otimes \mu_0 \otimes \mu_0 \otimes \ldots$ and $\theta = \mu_{t_1} \otimes \mu_{t_2} \otimes \mu_{t_3} \otimes \ldots$. If $t_k = \frac{1}{k^p}$, then find the values of p for which θ is singular to ν.

Problem 101. Let μ_t be the Ber(t) distribution. Let $\nu = \mu_{1/2} \otimes \mu_{1/2} \otimes \mu_{1/2} \otimes \ldots$ and $\theta = \mu_{t_1} \otimes \mu_{t_2} \otimes \mu_{t_3} \otimes \ldots$. If $t_k = \frac{1}{2} + \frac{c}{k^p}$ (assume $t_k < 1$), then find the values of p for which θ is singular to ν.

Problem 102. Prove that the statement of Kakutani's theorem for product measures remains valid if we define $a_n = \int_{\Omega_n} f_n^b d\nu_n$, where $0 < b < 1$. [*Note:* The usual statement (which is what we did in class) states this with $b = \frac{1}{2}$.]

Problem 103. Let μ, ν, θ be probability measures on $(\Omega, \mathcal{F})$. Suppose μ, ν are absolutely continuous to θ and $d\mu = f d\theta$ and $d\nu = g d\theta$. Show that the Lebesgue decomposition of μ w.r.t. ν is given by $d\mu = h d\nu + \gamma$ where $h = \frac{f}{g}\mathbf{1}_{g>0}$ and $\gamma(A) = \mu(A \cap \{g = 0\})$.

Problem 104. Let $(M_n)_{n\geq 0}$ be a martingale and let H be a predictable process. We know that $X_n = \sum_{k=1}^n H_k(M_k - M_{k-1})$ (with $X_0 = 0$) is a martingale, and hence X_n^2 is a sub-martingale.

(1) Find the Doob decomposition of X_n^2, i.e., write $X_n^2 = Y_n + A_n$ where Y is a martingale and A is a predictable increasing process.
(2) Specialise to the case when $M_n = \xi_1 + \ldots + \xi_n$, where $\xi_k \sim N(0,1)$ are i.i.d.

Problem 105. Let τ and σ be stopping times for $\mathcal{F}_\bullet$. Show that $A \cap \{\tau \leq \sigma\} \in \mathcal{F}_{\sigma\wedge\tau}$ for any $A \in \mathcal{F}_\tau$. Conclude that $\mathcal{F}_\tau \leq \mathcal{F}_\sigma$ if $\tau \leq \sigma$.

Problem 106. Let τ, τ' be finite stopping times for $\mathcal{F}_\bullet$. (a) Show that τ is $\mathcal{F}_\tau$ measurable. (b) Show that the event $\{\tau = \tau'\} \in \mathcal{F}_\tau \cap \mathcal{F}_{\tau'}$. (c) If X is a continuous process adapted to $\mathcal{F}_\bullet$, show that $X(\tau)$ is $\mathcal{F}_\tau$-measurable.

Problem 107. Let $\mathcal{F}^+$ be the right continuous extension of $\mathcal{F}$, i.e., $\mathcal{F}_t^+ = \cap_{h>0}\mathcal{F}_{t+h}$.

(1) Show that τ is an $\mathcal{F}^+$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for all t.
(2) Show that $(\mathcal{F}^+)_\tau = \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t \text{ for all } t\}$.
(3) Show that $(\mathcal{F}^+)_\tau = \mathcal{F}_\tau^+$ where the latter is defined as $\cap_{h>0}\mathcal{F}_{\tau+h}$ (which makes sense as $\tau + h$ is a stopping time for the original filtration for any $h > 0$).

Problem 108. Let $N(\cdot)$ be Poisson process with intensity $\lambda > 0$ on $\mathbb{R}_+$. Find a function $g_\theta(t)$ so that $e^{\theta(N(t)-\lambda t)-g_\theta(t)}$ is a martingale for any $\theta \in \mathbb{R}$.

Problem 109. We know that W_t and $t((W_t/\sqrt{t})^2 - 1)$ are martingales. Find similar martingales that are of the form $t^{d/2}Q_d(W_t/\sqrt{t})$ where Q_d is a polynomial of degree d, for $d = 3, 4$.

Problem 110. Let $(X_t)_{t\geq 0}$ be a $\mathcal{F}_\bullet$ martingale with RCLL sample path. Assume that it is bounded in L^p for some $p > 1$. Show that $X^* := \sup_t X_t$ is also in L^p and that $\mathbf{E}[|X^*|^p] \leq C_p \sup_t \mathbf{E}[|X_t|^p]$ for some $p > 1$. [*Remark:* Assume appropriate results from the discrete time theory]

Problem 111. Let $X = (X_t)_{t\geq 0}$ be a continuous martingale with $X_0 = 1$. Assume that $\tau = \inf\{t : X_t = 0\}$ is finite a.s. Find the distribution of the random variable $\sup\limits_{t\geq 0} X_{t\wedge\tau}$.

Problem 112. Let $f \in L^2_{\text{loc}}(\mathbb{R}_+)$ and let $X_t = \int_0^t f(s)dW(s)$ be the Wiener integral. Assume that it is defined so that the sample paths of X are continuous. Show that $\langle X\rangle_t = \int_0^t f(s)^2 ds$.

Problem 113.

Problem 114.

Problem 115.

Problem 116.

Problem 117.

Problem 118.