PROBLEMS IN PROBABILITY THEORY

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Disclaimer: I have collected or made up these problems from various books and other sources for the purpose of giving to students in a first course in measure theoretical probability. In many cases I have forgotten where I took them from or modified them from the original, and even those that I thought up myself are probably to be found in some book[1]. If anyone feels that specific problems require citation, I am happy to consider.

Problem 1. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If A_n is a countable sequence of subsets of Ω, the set $\limsup A_n$ (respectively $\liminf A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets A_n.

If $A_n \in \mathcal{F}$ for all n, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of A_ns and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

Problem 2. Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and A_n increase to A (respectively, decrease to A), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and A_ns increase to A, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

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[1]Here is a partial list of books that I have used at some time or another: Feller's *An introduction to probability theory and its applications: vol. 2*, Dudley's *Analysis and probability*, Khoshnevisan's *Probability*, Kallenberg's *Foundations of modern probability*, Durrett's *Probability: Theory and examples*, Uspensky's *Introduction to mathematical probability*, Pollard's *A user's guide to measure theoretic probability*, Williams' *Probability with martingales*, Chaumont and Yor's *Exercises in probability*. Possibly also Billingsley's *Probability and measure* and *Convergence of probability measures*, Grimmett and Stirzaker's *Probability and random processes* and K. R. Parthasarathy's *Probability measures on metric spaces*. I have also used from memory various problems from courses of Aldous and Peres that I sat in a long time ago (some are available on their homepages) as well as prelim/qualifier exams in multiple universities.

Problem 3. Suppose S is a π-system and is further closed under complements ($A \in S$ implies $A^c \in S$). Show that S is an algebra.

Problem 4. Let $\mathbf{P}$ be a p.m. on a σ-algebra $\mathcal{F}$ and suppose $S \subseteq \mathcal{F}$ be a π-system. If $A_k \in S$ for $k \leq n$, write $\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$ in terms of probabilities of sets in S.

Problem 5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbf{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Problem 6. Suppose $\sigma(S) = \mathcal{F}$ and $\mathbf{P}, \mathbf{Q}$ are two probability measure on $\mathcal{F}$. If $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in S$, is it necessarily true that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$? If yes, prove it. If not, give a counterexample.

Problem 7. Suppose $\mathcal{F} = \sigma(\mathcal{S})$ and $\mathbf{P}(A) \in \{0, \frac{1}{2}, 1\}$ for all $A \in \mathcal{S}$.

(1) If $\mathcal{S}$ is a π-system, show that $\mathbf{P}(A) \in \{0, \frac{1}{2}, 1\}$ for all $A \in \mathcal{F}$.

(2) If $\mathcal{S}$ is not a π-system, show that it is possible to have $\mathbf{P}(A) \notin \{0, \frac{1}{2}, 1\}$ for some $A \in \mathcal{F}$.

[*Note:* Think of other sets that can take the place of $\{0, \frac{1}{2}, 1\}$]

Problem 8. Let $\mathcal{F}$ be a sigma-algebra on $\mathbb{N}$ that is strictly smaller than the power set. Show that there exist $m \neq n$ such that elements of $\mathcal{F}$ do not separate m and n (i.e., any $A \in \mathcal{F}$ either contains both m, n or neither). Is the same conclusion valid if $\mathbb{N}$ is replaced by any set Ω?

Problem 9. Let $\mathbf{P}, \mathbf{Q}$ be two Borel probability measures on $\mathbb{R}^2$. If $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{S}$, can you conclude that $\mathbf{P} = \mathbf{Q}$. Deal with following cases:

(1) $\mathcal{S} = \{(a, b] \times (c, d] : a < b \text{ and } c < d\}$.

(2) $\mathcal{S} = \{(-\infty, b] \times (-\infty, d] : b, d \in \mathbb{R}\}$.

(3) $\mathcal{S} = \{(a, b] \times \mathbb{R} : a < b\} \cup \{\mathbb{R} \cup (c, d] : c < d\}$.

Problem 10. (1) Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}$. Show that $\mathcal{B}$ contains all closed sets, all compact sets, all intervals of the form $(a, b]$ and $[a, b)$.

(2) Show that there is a countable family $\mathcal{S}$ of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

(3) Let K be the 1/3-Cantor set. Show that $\mu_*(K) = 0$.

Problem 11. Show that each of the following collection of subsets of $\mathbb{R}^d$ generate the same sigma-algebra (which we call the Borel sigma-algebra).

(1) $\{(a, b] : a < b\}$.

(2) $\{[a, b] : a \leq b \text{ and } a, b \in \mathbf{Q}\}$.

(3) The collection of all open sets.

(4) The collection of all compact sets.

Problem 12. (1) Let X be an arbitrary set. Let S be the collection of all singletons in Ω. Describe $\sigma(S)$.

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b \text{ are real numbers}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra of $\mathbb{R}$.

(3) Suppose S is a collection of subsets of X and a, b are two elements of X such that any set in S either contains a and b both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both a and b or contains neither).

Problem 13. Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite }\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite.

(1) Show that $\mathcal{A}$ is an algebra and that μ is finitely additive on $\mathcal{A}$.

(2) Under what conditions does μ extend to a probability measure on $\mathcal{F} = \sigma(\mathcal{A})$?

Problem 14. On $\mathbb{N} = \{1, 2, \ldots\}$, let A_p denote the subset of numbers divisible by p. Describe $\sigma(\{A_p : p \text{ is prime}\})$ as explicitly as possible.

Problem 15. If $\mathcal{G} \subseteq \mathcal{F}$ are sigma algebras on Ω and $\mathcal{F}$ is countably generated (i.e., there is a countable collection of sets that generates the sigma-algebra), then is it necessarily true that $\mathcal{G}$ is countably generated?

Problem 16. Let $\mathcal{F} = \sigma\{A_i : i \in I\}$ where A_i, $i \in I$, are subsets of Ω. Given $B \in \mathcal{F}$, show that there is a countable subset $J \subseteq I$ such that $B \in \sigma\{A_i : i \in J\}$.

Problem 17. Let $E = \mathbb{R}^{[0,1]}$ be the space of all functions from $[0, 1]$ to $\mathbb{R}$. Let $\mathcal{F}$ be the cylinder sigma-algebra on E (cylinders are sets of the form $\{f \in E : f(t_i) \in B_i,\ 1 \leq i \leq n\}$ for some $t_1 < \ldots < t_n$ and some $B_i \in \mathcal{B}_{\mathbb{R}}$).

(1) Show that if $A \in \mathcal{F}$, then there is a countable set $\{t_i\} \subseteq [0, 1]$ such that membership in A is determined by the values of a function on the subset $\{t_i\}$.

(2) Show that $C[0,1]$ is not a measurable subset of E.

[*Remark:* The gist is that E is not a good space to model random functions]

Problem 18. On $C[0,1]$, show that the Borel sigma-algebra and the cylinder sigma-algebra are the same.

Here Borel sigma-algebra is w.r.t. the topology induced by the sup-norm metric. And cylinder sets are those of the form $\{f \in C[0,1] : f(t_i) \in B_i,\ 1 \le i \le n\}$ for some $0 \le t_1 < \ldots < t_n \le 1$ and some $B_i \in \mathcal{B}_{\mathbb{R}}$.

Problem 19. *Percolation* is a "random graph" described as follows: At each vertex of $\mathbb{Z}^d$, a p-coin is tossed to decide whether the vertex is open (heads) or closed (tails). We consider the random graph with all vertices, but edges only between adjacent pairs of open vertices.

(1) Write the probability space to capture this random experiment (at least write the sample space and sigma-algebra).

(2) Show that the set $A := \{x \in E : G_x \text{ has an infinite connected component}\}$ is measurable.

Problem 20. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces. If $T : X \to Y$ is a function, show that

(1) $\{T^{-1}B : B \in \mathcal{G}\}$ is a sigma algebra on X and

(2) $\{B \in \mathcal{G} : T^{-1}B \in \mathcal{F}\}$ is sigma-algebra on Y.

Problem 21. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces and $T : X \to Y$. Assume that $\mathcal{G} = \sigma(\mathcal{S})$ for some collection $\mathcal{S}$ of subsets of Y. Decide true or false:

If $T^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{S}$, then T is measurable.

Problem 22. Let $A_1, A_2, \ldots$ be a finite or countable partition of a non-empty set Ω (i.e., A_i are pairwise disjoint and their union is Ω). What is the σ-algebra generated by the collection of subsets $\{A_n\}$? What is the algebra generated by the same collection of subsets?

Problem 23. On $[0,1]$, let $\mathcal{A}$ be the algebra generated by finite unions of left-open, right-closed intervals and let $\mathcal{B}$ be the Borel sigma-algebra. Define $\mu : \mathcal{A} \to [0,1]$ by $\mu(A) = 1$ if $A \supseteq (0, \varepsilon)$ for some $\varepsilon > 0$ and $\mu(A) = 0$ otherwise.

Show that μ is a finitely additive measure on $\mathcal{A}$ but that it does not extend to a measure on $\mathcal{B}$. Why does this not contradict the Carathèodory extension theorem?

Problem 24. Let $X = [0,1]^{\mathbb{N}}$ be the countable product of copies of $[0,1]$. We define two sigma algebras of subsets of X.

(1) Define a metric on X by $d(x,y) = \sum_n |x_n - y_n| 2^{-n}$. Let $\mathcal{B}_X$ be the Borel sigma-algebra of (X,d). [**Note:** For those who know topology, it is better to define $\mathcal{B}_X$ as the Borel sigma algebra for the product topology on X. The point is that the metric is flexible. We can take many or other things (but not $d(x,y) = \sup_n |x_n - y_n|$!!). What matters is only the topology on X.]

(2) Let $\mathcal{C}_X$ be the sigma-algebra generated by the collection of all cylinder sets. Recall that cylinder sets are sets of the form $A = U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ where U_i are Borel subsets of $[0,1]$.

Show that $\mathcal{B}_X = \mathcal{C}_X$.

Problem 25. Let μ be the Lebesgue p.m. on the Cartheodary σ-algebra $\bar{\mathcal{B}}$ and let μ_* be the corresponding outer Lebesgue measure defined on all subsets of $[0,1]$. We say that a subset $N \subseteq [0,1]$ is a null set if $\mu_*(N) = 0$. Show that

$$\bar{\mathcal{B}} = \{B \cup N : B \in \mathcal{B} \text{ and } N \text{ is null}\}$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$.

[**Note:** The point of this exercise is to show how much larger is the Lebesgue σ-algebra than the Borel σ-algebra. The answer is, not much. Up to a null set, every Lebesgue measurable set is a Borel set. However, cardinality-wise, there is a difference. The Lebesgue σ-algebra is in bijection with $2^{\mathbb{R}}$ while the Borel σ-algebra is in bijection with $\mathbb{R}$.]

Problem 26. Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Say that a subset $N \subseteq \Omega$ is $\mathbf{P}$-null if there exists $A \in \mathcal{F}$ with $\mathbf{P}(A) = 0$ and such that $N \subseteq A$. Define $\mathcal{G} = \{A \cup N : A \in \mathcal{F} \text{ and } N \text{ is null}\}$.

(1) Show that $\mathcal{G}$ is a σ-algebra.

(2) For $A \in \mathcal{G}$, write $A = B \cup N$ with $b \in \mathcal{F}$ and a null set N, and define $\mathbf{Q}(A) = \mathbf{P}(B)$. Show that $\mathbf{Q}$ is well-defined, that $\mathbf{Q}$ is a probability measure on $\mathcal{G}$ and $\mathbf{Q}\Big|_{\mathcal{F}} = \mathbf{P}$.

[**Note:** $\mathcal{G}$ is called the $\mathbf{P}$-completion of $\mathcal{F}$. It is a cheap way to enlarge the σ-algebra and extend the measure to the larger σ-algebra. Another description of the extended σ-algebra is $\mathcal{G} = \{A \subseteq \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subseteq A \subseteq C \text{ and } \mathbf{P}(B) = \mathbf{P}(C)\}$. Combined with the previous problem, we see that the Lebesgue σ-algebra is just the completion of the Borel σ-algebra under the Lebesgue measure. However, note that completion depends on the probability measure (for a discrete probability measure on $\mathbb{R}$, the completion will be the power set σ-algebra!). For this reason, we prefer to stick to the Borel σ-algebra and not bother to extend it.]

Problem 27. Follow these steps to obtain Sierpinski's construction of a non-measurable set. Here μ_* is the outer Lebesgue measure on $\mathbb{R}$.

(1) Regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and choose a basis H (why is it possible?).

(2) Let $A_0 = H \cup (-H) = \{x : x \in H \text{ or } -x \in H\}$. For $n \geq 1$, define $A_n := A_{n-1} - A_{n-1}$ (may also write $A_n = A_{n-1} + A_{n-1}$ since A_0 is symmetric about 0). Show that $\bigcup_{n\geq 0}\bigcup_{q\geq 1}\frac{1}{q}A_n = \mathbb{R}$ where $\frac{1}{q}A_n$ is the set $\{\frac{x}{q} : x \in A_n\}$.

(3) Let $N := \min\{n \geq 0 : \mu_*(A_n) > 0\}$ (you must show that N is finite!). If A_N is measurable, show that $\cup_{n\geq N+1}A_n = \mathbb{R}$.

(4) Get a contradiction to the fact that H is a basis and conclude that A_N cannot be measurable.

[**Remark:** If you start with H which has zero Lebesgue measure, then $N \geq 1$ and $A := E_{N-1}$ is a Lebesgue measurable set such that $A + A$ is not Lebesgue measurable! That was the motivation for Sierpinski. To find such a basis H, show that the Cantor set spans $\mathbb{R}$ and then choose a basis H contained inside the Cantor set.]

Problem 28. For any $\varepsilon > 0$, show that there is a closed, totally disconnected set $A \subseteq [0,1]$ such that $\lambda(A) > 1 - \varepsilon$. Can you find such a set with $\lambda(A) = 1$? [*Note:* Totally disconnected means that the set contains no interval of positive length]

Problem 29. We saw that for a Borel probability measure μ on $\mathbb{R}$, the pushforward of Lebesgue measure on $[0,1]$ under the map $F_\mu^{-1} : [0,1] \to \mathbb{R}$ (as defined in lectures) is precisely μ. This is also a practical tool in simulating random variables. We assume that a random number generator gives us uniform random numbers from $[0,1]$. Apply the above idea to simulate random numbers from the following distributions (in matlab/mathematica or a program of your choice) a large number of times and compare the histogram to the actual density/mass function.

(1) Uniform distribution on $[a,b]$, (2) Exponential(λ) distribution, (3) Cauchy distribution, (4) Poisson(λ) distribution. What about the normal distribution?

Problem 30. Let $\Omega = X = \mathbb{R}$ and let $T : \Omega \to X$ be defined by $T(x) = x$. We give a pair of σ-algebras, $\mathcal{F}$ on Ω and $\mathcal{G}$ on X by taking $\mathcal{F}$ and $\mathcal{G}$ to be one of $2^{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{R}}$ or $\{\emptyset, \mathbb{R}\}$. Decide for each of the nine pairs, whether T is measurable or not.

Problem 31. (1) Define $T : \Omega \to \mathbb{R}^n$ by $T(\omega) = (\mathbf{1}_{A_1}(\omega), \ldots, \mathbf{1}_{A_n}(\omega))$ where $A_1, \ldots, A_n$ are given subsets of Ω. What is the smallest σ-algebra on Ω for which T becomes a random variable?

(2) Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and assume that $A_k \in \mathcal{F}$. Describe the push-forward measure $\mathbf{P} \circ T^{-1}$ on $\mathbb{R}^n$.

Problem 32. For $k \geq 0$, define the functions $r_k : [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ +1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n : [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under T_n

Problem 33.

(1) If $T : \mathbb{R}^n \to \mathbb{R}^m$, show that T is Borel measurable if it is (a) continuous or (b) right continuous or (c) lower semicontinuous or (d) non-decreasing (take $m = n = 1$ for the last one).

(2) If $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with the Lebesgue sigma-algebra, show that even if T is continuous, it need not be measurable! Just do this for $n = m = 1$.

Problem 34. Show that composition of random variables is a random variable. Show that real-valued random variables on a given $(\Omega, \mathcal{F})$ are closed under linear combinations, under multiplication, under countable suprema (or infima) and under limsup (or liminf) of countable sequences.

Problem 35. Let $\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{k/n}$ and let μ be the uniform p.m. on $[0,1]$. Show directly by definition that $d(\mu_n, \mu) \to 0$ as $n \to \infty$.

Problem 36. Show that each of the following is a metric that is equivalent to the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ (in the sense that $\mu_n \to \mu$ in one metric if and only if in the others).

(1) $\inf\{u > 0 : F_\mu(x + au\mathbf{1}) + bu \geq F_\nu(x),\ F_\nu(x + au\mathbf{1}) + bu \geq F_\mu(x)\ \forall x \in \mathbb{R}^d\}$ where $a, b > 0$ are fixed.

(2) $\inf\{u + v : u, v > 0 \text{ and } F_\mu(x + u\mathbf{1}) + v \geq F_\nu(x),\ F_\nu(x + u\mathbf{1}) + v \geq F_\mu(x)\ \forall x \in \mathbb{R}^d\}$.

Problem 37 (Change of variable for densities).

(1) Let μ be a p.m. on $\mathbb{R}$ with density f by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)dt$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push forward measure of μ under (a) $T(x) = x + a$ (b) $T(x) = bx$ (c) T is any increasing and differentiable function.

(2) If X has $N(\mu, \sigma^2)$ distribution, find the distribution of $(X - \mu)/\sigma$.

Problem 38. (1) Let $X = (X_1, \dots, X_n)$. Show that X is an $\mathbb{R}^d$-valued r.v. if and only if $X_1, \dots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \dots, \sigma(X_n)$?

(2) Let $X : \Omega_1 \to \Omega_2$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$ or vice versa. [**Extra!** If $Y : \Omega_1 \to \Omega_2$ is another r.v. which is measurable w.r.t. $\sigma(X)$ on Ω_1, then show that Y is a function of X].

Problem 39. The support of a probability measure μ on $\mathbb{R}^d$ is defined to be the smallest (inclusion-order) closed set C having $\mu(C) = 1$.

Show that the support is well-defined and is equal to $\{x \in \mathbb{R}^d : \mu(B(x, r)) > 0 \text{ for all } r > 0\}$.

Problem 40 (Lévy metric). (1) Show that the Lévy metric on $\mathcal{P}(\mathbb{R}^d)$ defined in class is actually a metric.

(2) Show that under the Lévy metric, $\mathcal{P}(\mathbb{R}^d)$ is a complete and separable metric space.

Problem 41. Show that each of the following is a metric that is equivalent to the Lévy metric (in the sense that $\mu_n \to \mu$ in one metric if and only if in the others).

(1) $\inf\{u > 0 : F_\mu(x + au\mathbf{1}) + bu \geq F_\nu(x),\ F_\nu(x + au\mathbf{1}) + bu \geq F_\mu(x)\ \forall x \in \mathbb{R}^d\}$ where $a, b > 0$ are fixed.

(2) $\inf\{u + v : u, v > 0 \text{ and } F_\mu(x + u\mathbf{1}) + v \geq F_\nu(x),\ F_\nu(x + u\mathbf{1}) + v \geq F_\mu(x)\ \forall x \in \mathbb{R}^d\}$.

Problem 42. Let μ, ν be probability measures on $\mathbb{R}$. Let $\mathcal{C}$ be the collection of all probability measures on $\mathbb{R}^2$ whose marginals are μ and ν. Show that $\mathcal{C}$ is tight in the space of probability measures on $\mathbb{R}^2$.

Problem 43 (Lévy-Prohorov metric). If (X, d) is a metric space, let $\mathcal{P}(X)$ denote the space of Borel probability measures on X. For $\mu, \nu \in \mathcal{P}(X)$, define

$$D(\mu, \nu) = \inf\{r \geq 0 : \mu(A_r) + r \geq \nu(A) \text{ and } \nu(A_r) + r \geq \mu(A) \text{ for all closed sets } A\}.$$

Here $A_r = \{y \in X : d(x, y) \leq r \text{ for some } x \in A\}$ is the closed r-neighbourhood of A.

(1) Show that D is a metric on $\mathcal{P}(X)$.

(2) When X is $\mathbb{R}^d$, show that this agrees with the definition of Lévy metric given in class (i.e., for any μ_n, μ, we have that $\mu_n \to \mu$ in both metrics or neither).

Problem 44 (**Lévy metric**)**.** Let $\mathcal{P}([-1,1]) \subseteq \mathcal{P}(\mathbb{R})$ be the set of all Borel probability measures μ such that $\mu([-1,1]) = 1$. For $\varepsilon > 0$, find a finite ε-net for $\mathcal{P}([-1,1])$. [*Note:* Recall that an ε-net means a subset such that every element of $\mathcal{P}([-1,1])$ is within ε distance of some element of the subset. Since $\mathcal{P}([-1,1])$ is compact, we know that a finite ε-net exists for all $\varepsilon > 0$.]

Problem 45. Consider $C[0,1]$ with the sup-norm metric and endowed with the Borel sigma-algebra. Let X is a $C[0,1]$-valued random variable on Ω (then X is called a *stochastic process* or a *random function*). Show that the following are random variables: (a) $U(\omega) = \max_{t\in[0,1]} X_t(\omega)$, (b) $V(\omega) = \lambda\{t \in [0,1] : X_t(\omega) > 0\}$ (here λ is the Lebesgue measure on $[0,1]$ as usual), (c) $L(\omega) = \max\{t \le 1 : X_t(\omega) = 0\}$.

Problem 46. Let $\mathcal{K}$ be the collection of non-empty compact subsets of $\mathbb{R}^d$. The Hausdorff metric on $\mathcal{K}$ is defined by $d(K,L) = \inf\{r > 0 : K^{(r)} \supseteq L,\ L^{(r)} \supseteq K\}$ where $K^{(r)} = \bigcup_{x\in K} B(x,r)$. Correspondingly, there is a Borel sigma-algebra on $\mathcal{K}$.

If X is a $\mathcal{K}$-valued random variable (called a *random set*), then show that the following are random variables: (a) $U(\omega) = \text{area}(X(\omega))$, (b) $V(\omega) = \mathbf{1}_{X(\omega)\cap A=\emptyset}$ where A is a fixed compact subset of $\mathbb{R}^d$, (c) $W(\omega) = \text{diameter}(X(\omega))$.

Problem 47. Using the Lévy metric, we can define a Borel sigma-algebra on $\mathcal{P}(\mathbb{R})$. Hence we can talk of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ taking values in $\mathcal{P}(\mathbb{R})$ (called *random probability measure*).

If $X_1, X_2, \ldots$ are real-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, show that $L_n(\omega) = \frac{1}{n}(\delta_{X_1(\omega)} + \ldots + \delta_{X_n(\omega)})$ is a random probability measure, for any $n \ge 1$.

Problem 48. On the probabiity space $([0,1], \mathcal{B}, \mu)$, for $k \ge 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \ge 1$, what is the distribution of X_n?

(2) For any fixed $n \ge 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the k^{th} digit in the binary expansion of t. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

Problem 49 (**Coin tossing space**)**.** Continuing with the previous example, consider the mapping $X : [0,1] \to \{0,1\}^{\mathbb{N}}$ defined by $X(t) = (X_1(t), X_2(t), \ldots)$. With the Borel σ-algebra on $[0,1]$ and the σ-algebra generated by cylinder sets on $\{0,1\}^{\mathbb{N}}$, show that X is a random variable and find the push-foward of the Lebesue measure under X.

Problem 50 (Equivalent conditions for weak convergence). Show that the following statements are equivalent to $\mu_n \xrightarrow{d} \mu$ (you may work in $\mathcal{P}(\mathbb{R})$).

(1) $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$ if F is closed.

(2) $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$ if G is open.

(3) $\limsup_{n\to\infty} \mu_n(A) = \mu(A)$ if $A \in \mathcal{F}$ and $\mu(\partial A) = 0$.

Problem 51. Fix $\mu \in \mathcal{P}(\mathbb{R})$. For $s \in \mathbb{R}$ and $r > 0$, let $\mu_{r,s} \in \mathcal{P}(\mathbb{R})$ be defined as $\mu_{r,s}(A) = \mu(rA + s)$ where $rA + s = \{rx + s : x \in A\}$. For which $R \subseteq (0, \infty)$ and $S \subseteq \mathbb{R}$ is it true that $\{\mu_{r,s} : r \in R,\ s \in S\}$ a tight family? [*Remark:* If not clear, just take μ to be the Lebesgue measure on $[0, 1]$.]

Problem 52. Let $\mu_n = \frac{2}{n(n+1)} \sum_{k=1}^{n} k\delta_{k/n}$. Then $\mu_n \xrightarrow{d} \mu$ as $n \to \infty$ for some $\mu \in \mathcal{P}(\mathbb{R})$ (which you must identify explicitly).

Problem 53. (1) Show that the family of Normal distributions $\{N(\mu, \sigma^2) : \mu \in \mathbb{R} \text{ and } \sigma^2 > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ and $B \subseteq (0, \infty)$ is the restricted family $\{N(\mu, \sigma^2) : \mu \in A \text{ and } \sigma^2 \in B\}$ tight?

Problem 54. (1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda > 0\}$ tight?

Problem 55. Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that F_μ is continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$. [*Restatement:* When F_μ is continuous, convergence to μ in Lévy-Prohorov metric also implies convergence in Kolmogorov-Smirnov metric.]

Problem 56. Show that the statement in the previous problem cannot be quantified. That is,

Given any $\varepsilon_n \downarrow 0$ (however fast) and $\delta_n \downarrow 0$ (however slow), show that there is some μ_n, μ with F_μ continuous, such that $d_{LP}(\mu_n, \mu) \le \varepsilon_n$ and $d_{KS}(\mu_n, \mu) \ge \delta_n$.

Problem 57. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and assume that $F_{\mu_n}(x) \to F_\mu(x)$ for all $x \in D$, a countable dense subset of $\mathbb{R}$. Does it follow that $\mu_n \xrightarrow{d} \mu$?

Problem 58. Consider the family of Normal distributions, $\{N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$. Show that the map $(\mu, \sigma^2) \to N(\mu, \sigma^2)$ from $\mathbb{R} \times \mathbb{R}_+$ to $\mathcal{P}(\mathbb{R})$ is continuous. (Complicated way of saying that if $(\mu_n, \sigma_n^2) \to (\mu, \sigma^2)$, then $N(\mu_n, \sigma_n^2) \stackrel{d}{\to} N(\mu, \sigma^2)$).

Do the same for other natural families if distributions, (1) Exp(λ), (2) Uniform$[a, b]$, (3) Bin(n, p) (fix n and show continuity in p), (4) Pois(λ).

Problem 59. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \stackrel{d}{\to} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

Problem 60. Which of the following sets are dense in $\mathcal{P}(\mathbb{R})$?

(1) The set of probability measures with finite support (those of the form $p_1\delta_{x_1} + \ldots + p_k\delta_{x_k}$).

(2) The set of probability measures having density.

(3) The set of probability measures with a symmetric density ($f(x) = f(-x)$).

(4) The set of probability measures having a bounded smooth density and all moments finite (densities $f \in C^\infty$ with $\int x^{2k} f(x)dx < \infty$ for all k).

(5) The set of probability measures with a unimodal density ($x \mapsto f(x)$ is increasing up to some x_0 and then decreasing).

Problem 61. Given a Borel p.m. μ on $\mathbb{R}$, show that it can be written as a convex combination $\alpha\mu_1 + (1-\alpha)\mu_2$ with $\alpha \in [0, 1]$, where μ_1 is a purely atomic Borel p.m and μ_2 is a Borel p.m with no atoms.

Problem 62. Let F be the CDF of a Borel probability measure μ on the line.

(1) Show that F is continuous at x if and only if $\mu(\{x\}) = 0$.

(2) Show that F can have at most countably many discontinuities.

(3) Show that given any countable set $\{x_1, x_2, \ldots\}$ and any number $p_1, p_2, \ldots$ such that $\sum_i p_i \le 1$, there is a probability measure whose CDF has a jump of magnitude p_i at x_i for each i, and no other discontinuities.

Problem 63. Let F be a CDF on $\mathbb{R}$. If $\sum_{x\in\mathbb{R}}(F(x) - F(x-))^2 = 1$, show that the measure is degenerate.

Problem 64. Let X be a random variable with distribution μ and X_n are random variables defined as follows. If μ_n is the distribution of X_n, in each case, show that $\mu_n \stackrel{d}{\to} \mu$ as $n \to \infty$.

(1) (Truncation). $X_n = (X \wedge n) \vee (-n)$.

(2) (Discretization). $X_n = \frac{1}{n}\lfloor nX \rfloor$.

Problem 65. Consider the space $X = [0,1]^{\mathbb{N}} := \{\mathbf{x} = (x(1), x(2), \ldots) : 0 \le x(i) \le 1 \text{ for each } i \in \mathbb{N}\}$. Define the metric $d(\mathbf{x}, \mathbf{y}) = \sup_i \frac{|x(i)-y(i)|}{i}$.

(1) Show that $\mathbf{x}_n \to \mathbf{x}$ in (X, d) if and only if $x_n(i) \to x(i)$ for each i, as $n \to \infty$.

[**Note:** What matters is this pointwise convergence criterion, not the specific metric. The resulting topology is called *product topology*. The same convergence would hold if we had defined the metric as $d(\mathbf{x}, \mathbf{y}) = \sum_i 2^{-i}|x(i) - y(i)|$ or $d(\mathbf{x}, \mathbf{y}) = \sum_i i^{-2}|x(i) - y(i)|$ etc., But not the metric $\sup_i |x(i) - y(i)|$ as convergence in this metric is equivalent to uniform convergence over all $i \in \mathbb{N}$].

(2) Show that X is compact.

[**Note:** What is this problem doing here? The purpose is to reiterate a key technique we used in the proof of Helly's selection principle!]

Problem 66. For $\varepsilon > 0$, find an ε-net for the space of probability measures supported on $[0, 1]$.

Problem 67. Find infinitely many measurable functions $f_n : [0,1] \to [0,1]$ such that $\lambda \circ f_n^{-1} = \lambda$. (In terms of random variables, if $U \sim \text{Unif}[0,1]$, then we want $f_n(U) \sim \text{Unif}[0,1]$ for all n.

Problem 68. Consider $([0,1], \mathcal{B})$ and let $T : [0,1] \to [0,1]$ be defined by $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$. Show that T preserves (i.e., $\mu \circ T^{-1} = \mu$) the probability measure with density $\frac{c}{1+x}$ (where $c = 1/\log 2$) on $[0,1]$. [*Note:* The sequence of integers $(\lfloor \frac{1}{x} \rfloor, \lfloor \frac{1}{T(x)} \rfloor, \lfloor \frac{1}{T(T(x))} \rfloor \ldots)$ is called the continued fraction expansion of x]

Problem 69. Recall the Cantor set $C = \bigcap_n K_n$ where $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, etc. In general, $K_n = \bigcup_{1 \le j \le 2^n} [a_{n,j}, b_{n,j}]$ where $b_{n,j} - a_{n,j} = 3^{-n}$ for each j.

(1) Let μ_n be the uniform probability measure on K_n. Describe its CDF F_n.

(2) Show that F_n converges uniformly to a CDF F.

(3) Let μ be the probability measure with CDF equal to F. Show that $\mu(C) = 1$.

Problem 70. Let $\mu \in \mathcal{P}(\mathbb{R})$.

(1) For any $n \ge 1$, define a new probability measure by $\mu_n(A) = \mu(n.A)$ where $n.A = \{nx : x \in A\}$. Does μ_n converge as $n \to \infty$?

(2) Let μ_n be defined by its CDF

$$F_n(t) = \begin{cases} 0 & \text{if } t < -n, \\ F(t) \text{ if } -n \le t < n, \\ 1 & \text{if } t \ge n. \end{cases}$$

Does μ_n converge as $n \to \infty$?

(3) In each of the cases, describe μ_n in terms of random variables. That is, if X has distribution μ, describe a transformation $T_n(X)$ that has the distribution μ_n.

Problem 71. Let $\mu_n = \frac{1}{Z_n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \delta_{k/n}$, where $f : [0,1] \to \mathbb{R}$ is a Borel measurable function. Show that $\mu_n \stackrel{d}{\to} \mu$ where $\mu = \lambda \circ f^{-1}$.

Work out the special case when $f(x) = x^p$, $p \in \mathbb{N}$.

Problem 72. In each case, decide if $\mu \ll \nu$ and if so, compute the Radon-Nikodym derivative.

(1) $\mu = \text{Bin}(n, p)$ and $\nu = \text{Bin}(n', p')$.

(2) $\mu = \text{Pois}(\lambda)$ and $\nu = \text{Pois}(\lambda')$.

(3) $\mu = N(\mu, \sigma^2)$ and $\nu = N(0, 1)$.

(4) $\mu = \text{Exp}(1)$ and $\nu = N(0, 1)$.

Problem 73. (Bernoulli convolutions) For any $\theta > 1$, define $X_\theta : [0,1] \to \mathbb{R}$ by $X_\theta(\omega) = \sum_{k=1}^{\infty} \theta^{-k} X_k(\omega)$. Check that X_θ is measurable, and define $\mu_\theta = \mu X_\theta^{-1}$. Show that for any $\theta > 2$, show that μ_θ is singular w.r.t. Lebesgue measure.

Problem 74. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let X, Y be bounded positive random variables on Ω and define two measures $\mu(A) = \mathbf{E}[X \mathbf{1}_A]$ and $\nu(A) = \mathbf{E}[Y \mathbf{1}_B]$ for any $A \in \mathcal{F}$ (i.e., $d\mu = X d\mathbf{P}$ and $d\nu = Y d\mathbf{P}$).

What should be the relationship between X and Y to ensure that (a) $\mu \perp \nu$? (b) $\mu \ll \nu$?

Problem 75. For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the above equivalence relation).

Problem 76. (1) Find a sequence of r.v.s X_n such that $\liminf \mathbf{E}[X_n] < \mathbf{E}[\liminf X_n]$.

(2) Find a sequence of r.v.s X_n such that $X_n \stackrel{a.s.}{\to} X$, $\mathbf{E}[X_n] = 1$, but $\mathbf{E}[X] = 0$.

Problem 77. Let $X_1, \ldots, X_n$ be random variables on a common probability space. Define $M_n = \max\{X_1, \ldots, X_n\}$ and $M_{n,\beta} = \frac{1}{\beta}\log(e^{\beta X_1} + \ldots + e^{\beta X_n})$. Show that $\mathbf{E}[M_{n,\beta}] \to \mathbf{E}[M_n]$ as $\beta \to \infty$. [*Remark*: M_n is got by applying a non-smooth function to X_is, but it can be approximated by $M_{n,\beta}$ which is a smooth function of X_is]

Problem 78. (**Alternate construction of Cantor measure**) Let $K_1 = [0, 1/3] \cup [2/3, 1]$, $K_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, etc., be the decreasing sequence of compact sets whose intersection is K. Observe that K_n is a union of 2^n intervals each of length 3^{-n}. Let μ_n be the p.m. which is the "renormalized Lebesgue measure" on K_n. That is, $\mu_n(A) := 3^n 2^{-n} \mu(A \cap K_n)$ for $A \in \mathcal{B}_{\mathbb{R}}$. Then each μ_n is a Borel p.m. Show that $\mu_n \stackrel{d}{\to} \mu$, the Cantor measure (which was defined differently in class).

Problem 79. (**A quantitative characterization of absolute continuity**) Suppose $\mu \ll \nu$. Then, show that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \varepsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

Problem 80. Let μ be a probability measure on $\mathbb{R}$ such that $\mu(a, b) \le M(b - a)$ for some $M < \infty$ and all $a < b$. Show that μ has a bounded density.

Problem 81. If $\mu \in \mathcal{P}(\mathbb{R}^d)$ has density f, show that $\int_{\mathbb{R}^d} g(x) d\mu(x) = \int_{\mathbb{R}^d} g(x) f(x) dx$ (the latter is integral w.r.t. Lebesgue measure on $\mathbb{R}^d$). [*Note:* A more general question is in Problem 89 below. The point is that if $X \sim \mu$, then we can compute $\mathbf{E}[g(X)]$ directly from the density of X]

Problem 82. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and let $\theta = \frac{1}{2}\mu + \frac{1}{2}\nu$.

(1) Show that $\mu \ll \theta$ and $\nu \ll \theta$.

(2) If $\mu \perp \nu$, describe the Radon Nikodym derivative of μ w.r.t. θ.

Problem 83. Let μ_i, $i \in I$, be probability measures on $\mathbb{R}$.

(1) If I is countable, show that there is a $\theta \in \mathcal{P}(\mathbb{R})$ such that $\mu_i \ll \theta$ for all i.

(2) If I is uncountable, show that the conclusion of the first part may fail.

Problem 84. Let μ and ν be Borel probability measures on $\mathbb{R}$. Suppose there exists a probability measure θ on $\mathbb{R}^2$ having marginals $\theta \circ \Pi_1^{-1} = \mu$ and $\theta \circ \Pi_2^{-1} = \nu$ such that $\theta\{(x, x) : x \in \mathbb{R}\} > 0$. Then show that μ and ν cannot be singular.

[In the language of random variables, the hypothesis says that we can couple $X \sim \mu$ and $Y \sim \nu$ such that $X = Y$ with positive probability.]

Problem 85. (Requires product measure) Decide true or false and justify. Take μ_i, ν_i to be probability measures on $(\Omega_i, \mathcal{F}_i)$.

(1) If $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$, then $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$.

(2) If $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$, then $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$.

Problem 86. (Requires knowledge of infinite product measure) Decide whether the following pairs of measures on $\mathbb{R}^{\mathbb{N}}$ (with the product sigma-algebra) are singular or absolutely continuous to one another? In each case, $\mu = \mu_1 \otimes \mu_2 \otimes \ldots$ and $\nu = \nu_1 \otimes \nu_2 \otimes \ldots$.

(1) $\mu_i = \text{Ber}(1/2)$ and $\nu_i = \text{Ber}(p)$ for all i, for some $p \in [0, 1]$.

(2) $\mu_i = N(0, 1)$ and $\nu_i = N(\alpha, 1)$ for all i, where $\alpha \in \mathbb{R}$.

(3) $\mu_i = \text{Ber}(1/2)$ and $\nu_i = \text{Ber}(p_i)$ for all i, where $p_i = 1/2$ for $i > 100$.

(4) $\mu_i = N(0, 1)$ and $\nu_i = N(\alpha_i, 1)$ for all i, where $\alpha_i = 0$ for $i > 1000$.

Problem 87. Suppose $f : [a, b] \to \mathbb{R}$ is a Borel measurable function. Then, show that $g(x) := \int_0^x f(u)du$ is a continuous function on $[0, 1]$. [**Note:** It is in fact true that g is differentiable at almost every x and that $g' = f$ *a.s.*, but that is a more sophisticated fact, called *Lebesgue's differentiation theorem*. In this course, we only need Lebesgue integration, not differentiation. The latter may be covered in your measure theory class].

Problem 88. (**Differentiating under the integral**). Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, satisfy the following assumptions.

(1) $x \to f(x, \theta)$ is Borel measurable for each θ.

(2) $\theta \to f(x, \theta)$ is continuously differentiable for each x.

(3) $f(x, \theta)$ and $\frac{\partial f}{\partial \theta}(x, \theta)$ are uniformly bounded functions of (x, θ).

Then, justify the following "differentiation under integral sign" (including the fact that the integrals here make sense).

$$\frac{d}{d\theta} \int_a^b f(x, \theta)dx = \int_a^b \frac{\partial f}{\partial \theta}(x, \theta)\, dx$$

[**Hint:** Remember that derivative is the limit of difference quotients, $h'(t) = \lim_{\varepsilon \to 0} \frac{h(t+\varepsilon)-h(t)}{\varepsilon}$.

Problem 89. Let $X \geq 0$ be a r.v on $(\Omega, \mathcal{F}, \mathbf{P})$ with $0 < \mathbf{E}[X] < \infty$. Then, define $\mathbf{Q}(A) = \mathbf{E}[X\mathbf{1}_A]/\mathbf{E}[X]$ for any $A \in \mathcal{F}$. Show that $\mathbf{Q}$ is a probability measure on $\mathcal{F}$. Further, show that for any bounded random variable Y, we have $\mathbf{E}_{\mathbf{Q}}[Y] = \frac{\mathbf{E}[YX]}{\mathbf{E}[X]}$.

Problem 90. If μ and ν are Borel probability measures on the line with continuous densities f and g (respectively) w.r.t. Lebesgue measure. Under what conditions can you assert that μ has a density w.r.t ν? In that case, what is that density?

Problem 91. For $p = 1, 2, \infty$, check that $\|X-Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the equivalence relation $X \sim Y$ if $\mathbf{P}(X = Y) = 1$).

Problem 92. If X is an integrable random variable, show that there are bounded random variables X_n such that $\mathbf{E}[|X_n - X|] \to 0$ as $n \to \infty$.

Problem 93. Let $0 < p < q$.

(1) If $X \in L^q$, show that $X \in L^p$.

(2) If $\mathbf{E}[|X_n|^q] \to 0$ show that $\mathbf{E}[|X_n|^p] \to 0$.

Problem 94. Find integrable random variables X_n, X for each of the following situations.

(1) $X_n \to X$ a.s. but $\mathbf{E}[X_n] \not\to \mathbf{E}[X]$.

(2) $X_n \to X$ a.s. and $\mathbf{E}[X_n] \to \mathbf{E}[X]$ but there is no dominating integrable random variable Y for the sequence $\{X_n\}$.

[*Remark:* That is, the domination condition cannot be removed but can perhaps be weakened.]

Problem 95. Let X be a non-negative random variable.

(1) Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}dt$ (in particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \ge n)$).

(2) Show that $\mathbf{E}[X^p] = \int_0^\infty pt^{p-1}\mathbf{P}\{X \ge t\}dt$ for any $p > 0$.

(3) Show that $\mathbf{E}[e^{\theta X}] = \int \theta e^{\theta t}\mathbf{P}\{X \ge t\}dt$ for any $\theta \in \mathbb{R}$.

Problem 96. For any integrable random variable X having CDF F, show that

$$\mathbf{E}[X] = \int_0^\infty (1 - F(x) + F(-x))dx.$$

Problem 97. Let X be a non-negative random variable. If $\mathbf{E}[X]$ is finite, show that $\sum_{n=1}^\infty \mathbf{P}\{X \ge an\}$ is finite for any $a > 0$. Conversely, if $\sum_{n=1}^\infty \mathbf{P}\{X \ge an\}$ is finite for some $a > 0$, show that $\mathbf{E}[X]$ is finite.

Problem 98. Decide true or false: A random variable X has finite kth moment if and only if $Q(X)$ has finite expectation for some polynomial of degree k.

Problem 99. Suppose $\mu \in \mathcal{P}(\mathbb{R})$ satisfies $\mu(a,b) \le C(b-a)^p$ for all $a < b$ for some $p > 0$ and $C < \infty$. Assume that μ is compactly supported (so that the issues here are not at what happens near ∞)

(1) Show that $\frac{1}{|x-a|^q}$ is integrable w.r.t. μ for any $q < p$.

(2) Show that $\log|x-a|$ is integrable w.r.t. μ for any $a \in \mathbb{R}$.

[*Remark*: Note that $p \le 1$ necessarily]

Problem 100. Is there any probability distribution μ on $\mathbb{R}$ such that for every $a \in \mathbb{R}$, the function $\frac{1}{x-a}$ is integrable w.r.t. μ? [*Equivalent form*: Does there exist a random variable X such that $\mathbf{E}\left[\frac{1}{|X-a|}\right] < \infty$ for all $a \in \mathbb{R}$?]

Problem 101. Let X be an random variable with mean μ and finite variance σ^2 and a median M (there can be multiple medians). Show that $|\mu - M| \le \sqrt{2}\sigma$.

Problem 102. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, convex and bijective. Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and for a random variable X, define $\|X\|_\Psi := \inf\{b > 0 : \mathbf{E}[\Psi(|X|/b)] \le 1\}$ (the infimum of empty set is $+\infty$). Let $L^\Psi = \{X : \|X\|_\Psi < \infty\}$.

(1) Show that $\|\cdot\|_\Psi$ is a pseudo-norm on L^Ψ (it becomes a norm on the space of equivalence classes).

(2) What choice of Ψ gives L^p norm for $1 \le p < \infty$?

Problem 103. Show that the values $\mathbf{E}[f \circ X]$ as f varies over the class of all smooth (infinitely differentiable), compactly supported functions determine the distribution of X.

Problem 104. (i) Express the mean and variance of of $aX + b$ in terms of the same quantities for X (a, b are constants).

(ii) Show that $\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$.

Problem 105. Compute mean, variance and moments (as many as possible!) of the Normal$(0,1)$, exponential(1), Beta(p,q) distributions.

Problem 106. Consider a probability density f on $\mathbb{R}$ that is symmetric and log-concave, i.e., $f(x) = e^{-\varphi(x)}$ where φ is an even convex function.

(1) Show that f has moments of all orders.

(2) Show that there are universal constants a, b such that $a \leq f(0)\sigma^2 \leq b$ where σ^2 is the variance.

Problem 107. If μ and ν are probability measures on a finite set $\mathcal{A}$, then the *relative entropy* of μ w.r.t. ν is defined as $D(\mu\|\nu) = \sum_{a\in\mathcal{A}} \mu(a)\log\frac{\mu(a)}{\nu(a)}$. The quantity $H(\mu) := \sum_{a\in\mathcal{A}} \mu(a)\log\frac{1}{\mu(a)}$ is called the *entropy* of μ.

(1) Show that $D(\mu\|\nu) \geq 0$ with equality if and only if $\mu = \nu$.

(2) Show that $0 \leq H(\mu) \leq \log|\mathcal{A}|$. When are the inequalities attained?

[*Clarification:* When $\mu(a) = 0$ the sumand is taken to be 0 but when $\mu(a) > 0$ but $\nu(a) = 0$, it is taken to be $+\infty$.].

Problem 108. For two probability densities f, g on $\mathbb{R}$, the *relative entropy* of the first with respect to the second is defined as $D(f\|g) = \int_{\mathbb{R}} f(x)\log\frac{f(x)}{g(x)}dx$. Show that $D(f\|g) \geq 0$ with equality if and only if $f = g$ a.e. [*Clarification:* When $f(x) = 0$ the integrand is taken to be 0 but when $f(x) > 0$ but $g(x) = 0$, it is taken to be $+\infty$.]

Problem 109. Find $D(f\|g)$ if

(1) f is the $N(\mu, \sigma^2)$ density and g is the $N(0, 1)$ density.

(2) f is the standard Cauchy density and g is the $N(0, 1)$ density.

Problem 110. Let $\theta_p = p\delta_1 + (1-p)\delta_0$ for $0 \leq p \leq 1$. Let $\mu_{n,p} = \otimes_{k=1}^n \theta_p$. Find $D(\mu_{n,p}\|\mu_{n\frac{1}{2}})$ and analyse what happens as $n \to \infty$.

Problem 111. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be measures without atoms (so their CDFs are continuous). Let $D = \{(x, x) : x \in \mathbb{R}\}$ be the diagonal. Show that $(\mu \otimes \nu)(D) = 0$.

Problem 112. If X, Y are independent random variables with continuous distributions, then $\mathbf{P}\{X = Y\} = 0$. [*Remark:* This is a restatement of the previous exercise in terms of random variables]

Problem 113. Let $X \sim \mu$ and $Y \sim \nu$ be independent random variables and let θ be the distribution of $X + Y$. Decide True/False and justify.

(1) If μ and ν are both discrete, then so is θ.

(2) If μ and ν are both absolutely continuous, then so is θ.

(3) If μ is discrete and ν is absolutely continuous, then θ is absolutely continuous.

Problem 114. Drop the assumption of independence in the previous problem (answers may change!).

Problem 115. Let $X_1, \ldots, X_n$ be positive random variables on a common probability space. Show that

$$\mathbf{E}[\max\{X_1, \ldots, X_n\}] \le \sum_{k=1}^{n} \mathbf{E}[X_k] \le \mathbf{E}[\max\{X_1, \ldots, X_n\}] + \sum_{i<j} \mathbf{E}[\min\{X_i, X_j\}].$$

Problem 116. (1) If $X_n \ge 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

(2) If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{|X|>A}] \to 0$ as $A \to \infty$.

Problem 117. (1) Suppose (X, Y) has a continuous density $f(x, y)$. Find the density of X/Y. Apply to the case when (X, Y) has the *standard bivariate normal distribution* with density $f(x, y) = (2\pi)^{-1} \exp\{-\frac{x^2+y^2}{2}\}$.

(2) Find the distribution of $X + Y$ if (X, Y) has the standard bivariate normal distribution.

(3) Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find the density of (U, V).

Problem 118. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$. Show that $\mu_n \stackrel{d}{\to} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b(\mathbb{R})$. What if we only assume $\int f d\mu_n \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^n)$ - can we conclude that $\mu_n \stackrel{d}{\to} \mu$?

Problem 119. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$ having densities f_n, f with respect to Lebesgue measure. If $f_n \to f$ a.e. (w.r.t. Lebesgue measure), show that $\mu_n \stackrel{d}{\to} \mu$.

Problem 120 (Moment matrices). Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \ge 1$, show that the matrix $(\alpha_{i+j})_{0 \le i,j \le n}$ is non-negative definite. [**Suggestion:** First solve $n = 1$].

Problem 121. Let $X \ge 0$ and let $m_p = \mathbf{E}[X^p]$. If $1 \le p_1 < p_2 < p_3$, show that $m_{p_2} \le m_{p_1}^a \times m_{p_3}^b$ for some a, b that depend on p_is but not on the distribution of X.

Problem 122. Let X be a non-negative random variable with all moments (i.e., $\mathbf{E}[X^p] < \infty$ for all $p < \infty$). Show that $\log \mathbf{E}[X^p]$ is a convex function of p.

Problem 123. (1) Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$. Assume that μ_n has density f_n and μ has density f w.r.t Lebesgue measure on $\mathbb{R}^n$. If $f_n(t) \to f(t)$ for all t, then show that $\mu_n \stackrel{d}{\to} \mu$.

(2) Show that $N(\mu_n, \sigma_n^2) \stackrel{d}{\to} N(\mu, \sigma)$ if and only if $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$.

Problem 124. (1) Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\alpha', 1)$ be independent random variables on a common probability space. Find the distribution of $\frac{X}{X+Y}$.

(2) If U, V are independent and have uniform([0,1]) distribution, find the distribution of $U + V$.

Problem 125. Let $\Omega = \{1, 2, \ldots, n\}$. For a probability measure $\mathbf{P}$ on Ω, we define it "entropy" $H(\mathbf{P}) := -\sum_{k=1}^{n} p_k \log p_k$ where $p_k = \mathbf{P}\{k\}$ and it is understood that $x \log x = 0$ if $x = 0$. Show that among all probability measures on Ω, the uniform probability measure (the one with $p_k = \frac{1}{n}$ for each k) is the unique maximizer of entropy.

Problem 126. (1) If $\mu_n \ll \nu$ for each n and $\mu_n \stackrel{d}{\to} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each n and $\mu_n \stackrel{d}{\to} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose X, Y are independent (real-valued) random variables with distribution μ and ν respectively. If μ and ν are absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X + Y$ is also absolutely continuous w.r.t Lebesgue measure.

Problem 127. Suppose $\{\mu_\alpha : \alpha \in I\}$ and $\{\nu_\beta : \alpha \in J\}$ are two families of Borel probability measures on $\mathbb{R}$. If both these families are tight, show that the family $\{\mu_\alpha \otimes \nu_\beta : \alpha \in I,\ \beta \in J\}$ is also tight.

Problem 128. Let X be a non-negative random variable. If $\mathbf{E}[X] \leq 1$, then show that $\mathbf{E}[X^{-1}] \geq 1$.

Problem 129. Suppose X, Y are independent random variables and $X + Y$ has finite expectation. Then show that X has finite expectation. [*Hint:* Assume that Y has symmetric distribution to get a possibly simpler version of the problem]

Problem 130. On the probabiity space $([0,1], \mathcal{B}, \mu)$, for $k \geq 1$, define the functions

$$X_k(t) := \begin{cases} 0 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j}{2^k}, \frac{2j+1}{2^k}). \\ 1 & \text{if } t \in \bigcup_{j=0}^{2^{k-1}-1} [\frac{2j+1}{2^k}, \frac{2j+2}{2^k}) \text{ or } t = 1. \end{cases}$$

(1) For any $n \geq 1$, what is the distribution of X_n?

(2) For any fixed $n \geq 1$, find the joint distribution of $(X_1, \ldots, X_n)$.

[**Note:** $X_k(t)$ is just the k^{th} digit in the binary expansion of t. Dyadic rationals have two binary expansions, and we have chosen the finite expansion (except at $t = 1$)].

Problem 131. If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

Problem 132 (A quantitative characterization of absolute continuity). Suppose $\mu \ll \nu$. Then, show that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \varepsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

Problem 133. For $\mu, \nu \in \mathcal{P}(\mathbb{R})$, a budding probabilist asserts that $\nu \ll \mu$ provided $\nu(I) = 0$ for all intervals for which $\mu(I) = 0$. Will he or she bud or wither? What if intervals are replaced by compact sets?

Problem 134. Let $Z_1, \ldots, Z_n$ be i.i.d $N(0,1)$ and write $\mathbf{Z}$ for the vector with components $Z_1, \ldots, Z_n$. Let A be an $m \times n$ matrix and let μ be a vector in $\mathbb{R}^m$. Then the m-dimensional random vector $\mathbf{X} = \mu + A\mathbf{Z}$ is said to have distribution $N_m(\mu, \Sigma)$ where $\Sigma = AA^t$ ('Normal distribution with mean vector μ and covariance matrix Σ').

(1) If $m \leq n$ and A has rank m, show that $\mathbf{X}$ has density $(2\pi)^{-\frac{m}{2}} \exp\{-\frac{1}{2}\mathbf{x}^t A^{-1}\mathbf{x}\}$ w.r.t Lebesgue measure on $\mathbb{R}^m$. In particular, note that the distribution depends only on μ and AA^t. (**Note:** If $m > n$ or if rank$(A) < m$, then satisfy yourself that $\mathbf{X}$ has no density w.r.t Lebesgue measure on $\mathbb{R}^m$ - you do not need to submit this).

(2) Check that $\mathbf{E}[X_i] = \mu_i$ and $\text{Cov}(X_i, X_j) = \Sigma_{i,j}$.

(3) What is the distribution of (i) $(X_1, \ldots, X_k)$, for $k \leq n$? (ii) $B\mathbf{X}$, where B is a $p \times m$ matrix? (iii) $X_1 + \ldots + X_m$?

Problem 135. (1) If X, Y are independent random variables, show that $\text{Cov}(X, Y) = 0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1, X_2, \ldots$ such that $\text{Cov}(X_i, X_j) = 0$ for any $i \neq j$ but such that X_i are not independent.

(3) Suppose $(X_1, \dots, X_m)$ has (joint) normal distribution (see the first question). If $\text{Cov}(X_i, X_j) = 0$ for all $i \le k$ and for all $j \ge k+1$, then show that $(X_1, \dots, X_k)$ is independent of $(X_{k+1}, \dots, X_m)$.

Problem 136. Decide whether the following are true or false and explain why.

(1) If X is independent of itself, X is constant a.s.

(2) If X is independent X^2 then X is a constant a.s.

(3) If $X, Y, X+Y$ are independent, then X and Y are constants a.s.

(4) If X and Y are independent and also $X+Y$ and $X-Y$ are independent, then X and Y must be constants a.s.

Problem 137. If $X \sim \text{Exp}(1)$, show that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are independent. Give examples of distributions other than Exponential for which the same independence holds.

Problem 138. Let X, Y be independent random variables such that $\mathbf{P}\{X \le Y\} = 1$. Show that there exists some $t \in \mathbb{R}$ such that $\mathbf{P}\{X \le t\} = 1$ and $\mathbf{P}\{Y \ge t\} = 1$.

Problem 139. (1) Suppose $2 \le k < n$. Give an example of random variables $X_1, \dots, X_n$ such that any subset of k of these random variables are independent but no subset of $k+1$ of them is independent.

(2) Suppose $(X_1, \dots, X_n)$ has a multivariate Normal distribution. Show that if X_i are pairwise independent, then they are independent.

Problem 140. Let $\Omega = \{0,1\}^n$ with its power set sigma-algebra and uniform distribution $\mathbf{P}$. Show that it is not possible to define $n+1$ non-constant random variables that are independent. [*Hint:* First show that it is not possible to get $n+1$ independent $\text{Ber}(p_i)$ random variables with $0 < p_i < 1$]

Problem 141. Show that it is not possible to define uncountably many independent Ber(1/2) random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

Problem 142. Let $\Omega = \{1, 2, \dots, n\}$ with the power set sigma-algebra and uniform probability measure. Let $X_p(k) = \mathbf{1}_{p \text{ divides } k}$. Are X_2 and X_3 independent? [*Note:* The answer may depend on n.]

Problem 143. Suppose $N \geq 2$ distinguishable balls are thrown into $m \geq 2$ labelled bins, uniformly at random.

(1) Show that X_is are not independent.

(2) If N is itself a random number from Pois(λ) distribution (and then the assignment of balls to bins is independent of N), show that X_is are independent and find their distributions.

Problem 144. For a real valued random variable X, its *concentration function* is defined as $Q_X(t) = \sup\{\mathbf{P}\{X \in [a, a+t] : a \in \mathbb{R}\}$, for $t \geq 0$ (so $Q_X(0)$ is the largest atom size in the distribution of X). If X, Y are independent and $Z = X + Y$, show that $Q_{X+Y}(t) \leq Q_X(t)$ for all $t \geq 0$.

Problem 145. Let U_n, V_n, $n \geq 1$ be i.i.d. Unif$[0,1]$ random variables. Let $X_k = \mathbf{1}_{U_k \leq V_1}$ and $Y_k = \mathbf{1}_{U_k < V_k}$ and $Z_k = \mathbf{1}_{U_k < \min_{j \leq k} V_j}$.

Which of the collections $\{X_n\}$, $\{Y_n\}$, $\{Z_n\}$ are (A) Independent? (B) Identically distributed?

Problem 146. Pick a permutation $\Pi \in \mathcal{S}_n$ uniformly at random. Let X_k be the indicator of the event that k is the smallest element in its cycle (in the cycle decomposition of Π). Show that $X_1, \ldots, X_n$ are independent and that $X_k \sim \text{Ber}(1/k)$.

Problem 147. Pick a permutation $\Pi \in \mathcal{S}_n$ uniformly at random. Let Y_k be the number of $j < k$ for which $\Pi(k) < \Pi(j)$. Show that $Y_1, \ldots, Y_n$ are independent and that $Y_k \sim \text{Uniform}\{0, 1, \ldots, k-1\}$.

Problem 148. Let X_i, $i \geq 1$ be random variables on a common probability space. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable function (with product sigma algebra on $\mathbb{R}^{\mathbb{N}}$ and Borel sigma algebra on $\mathbb{R}$) and let $Y = f(X_1, X_2, \ldots)$. Show that the distribution of Y depends only on the joint distribution of $(X_1, X_2, \ldots)$ and not on the original probability space. [**Hint:** We used this to say that if X_i are independent Bernoulli random variables, then $\sum_{i \geq 1} X_i 2^{-i}$ has uniform distribution on $[0,1]$, irrespective of the underlying probability space.]

Problem 149. Let $(\Omega_1, \mathcal{F}_1, \mu)$, $(\Omega_2, \mathcal{F}_2, \nu)$ be probability spaces and let θ be a probability measure on $(\Omega = \Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. We write $z \in \Omega$ as $z = (x, y)$ (i.e., $x = \Pi_1(z)$ and $y = \Pi_2(z)$).

(1) Show that θ has marginals μ and ν if and only if,

$$\int_{\Omega} (f(x) + g(y)) d\theta(z) = \int_{\Omega_1} f d\mu + \int_{\Omega_2} g d\nu.$$

for every f, g bounded random variables on Ω_1 and Ω_2 respectively.

(2) Show that $\theta = \mu \otimes \nu$ if and only if

$$\int_{\Omega} f(x)g(y)d\theta(z) = \left(\int_{\Omega_1} f d\mu\right) \times \left(\int_{\Omega_2} g d\nu\right)$$

for every f, g bounded random variables on Ω_1 and Ω_2 respectively.

Problem 150. Let X be a random variable taking values in $\{0, 1, \ldots, n\}$ with $p_k = \mathbf{P}\{X = k\}$. Let $P(t) = p_0 + p_1 t + \ldots + p_n t^n$ be the generating function of X. Show that the following are equivalent:

(1) All roots of P are real.

(2) X has the same distribution as a sum of n independent (not necessarily identical) Bernoulli random variables.

Problem 151. Suppose $(X_1, \ldots, X_n)$ has density f (w.r.t Lebesgue measure on $\mathbb{R}^n$).

(1) If $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions g_k, $k \le n$. Then show that $X_1, \ldots, X_n$ are independent. (Don't assume that g_k is a density!)

(2) If $X_1, \ldots, X_n$ are independent, then $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1, \ldots, g_n$.

Problem 152. (1) Let S be the set of all $x \in [0, 1]$ whose base b-expansion contains all the digits $0, 1, \ldots, b-1$, for every $b \in \{2, 3, 4 \ldots\}$. Show that $\lambda(S) = 1$, where λ is the Lebesgue measure on $[0, 1]$.

(2) Let S be the set of all points in $\mathbb{R}^2$ that can be written as a convex combination of two rational points (a rational point is one whose co-ordinates are all rational numbers). Show that S has zero Lebesgue measure.

Problem 153. Let $X = (X_{i,j})_{i,j \le n}$ where $X_{i,j}$ are i.i.d. $N(0, 1)$ random variables and let $A = (X + X^t)/\sqrt{2}$ (a random symmetric matrix). Let $\lambda_1 \le \ldots \le \lambda_n$ be the eigenvalues of A (repeated with multiplicity). Let $J \sim \text{Unif}[n]$ independent of $X_{i,j}$s. Let $\lambda = \lambda_J$ (a uniformly randomly chosen eigenvalue of A).

(1) Show that $\mathbf{E}[\lambda^p] = 0$ if p is odd.

(2) Show that $\mathbf{E}[\lambda^2] = 1$ and $\mathbf{E}[\lambda^4] = 2$.

[*Much more difficult:* Find $\mathbf{E}[\lambda^p]$ for larger even numbers, $p = 6, 8, \ldots$]

Problem 154. Let X, Y be random variables on a common probability space. Assume that both X and Y have finite variance.

(1) Show that $\mathbf{E}[(X-a)^2]$ is minimized uniquely at $a = \mathbf{E}[X]$.

(2) Find values of a, b that minimize $f(a,b) = \mathbf{E}\left[(Y-a-bX)^2\right]$. Are they unique?

(3) Suppose $\mathbf{P}(X=k) = \frac{1}{10}$ for $k = 1, 2 \ldots, 10$. At what value(s) of a is $\mathbf{E}[|X-a|]$ minimized? Is the minimizer unique?

Problem 155. Let $X_1, \ldots, X_n$ be i.i.d. random variables with a common distribution function F. Assume that F has a density f. Let $X_{(k)}$ be the kth *order statistic*, i.e., the kth smallest among $X_1, \ldots, X_n$ (e.g., $X_{(n)} = \max_{i \le n} X_i$). Show that $X_{(k)}$ has a density given by

$$g_k(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1}(1-F(x))^{n-k} f(x).$$

Problem 156. If X_i are i.i.d. Unif$[0,1]$, identify the distribution of $X_{(k)}$, the kth order statistic. Find $\mathbf{E}[X_{(k)}]$ and $\mathrm{Var}(X_{(k)})$.

Problem 157. There are n machines and n jobs. Machine i can do job j at a cost of $\xi_{i,j}$. The optimal assignment is to pair machines with jobs so that the total cost is minimized. Let $\xi_{i,j}$ be i.i.d. Exp(1) random variables and let $\mathcal{C}_n$ be the cost of the optimal assignment. Show that $\mathbf{E}[\mathcal{C}_n] = O(\log n)$. [*Note:* In fact, it is known that $\mathbf{E}[\mathcal{C}_n] = 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ which remains bounded! But that is more involved. Look up *random assignment problem* to know more]

Problem 158. Among all $n!$ permutations of $[n]$, pick one at random with uniform probability. Show that the probability that this random permutation has no fixed points is at most $\frac{1}{2}$ for any n.

Problem 159. Suppose each of $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\left| \frac{N_n}{n} - e^{-\lambda} \right| > \delta \right) \to 0$$

Problem 160. Let X_n be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of f is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

Problem 161. (1) Let X be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X-a|]$? [**Hint:** First consider X that takes finitely many values with equal probability each].

Problem 162. If X is a positive random variable, show that $\mathbf{E}[X^p]^{\frac{1}{p}}$ is increasing in $p \in [0, \infty)$.

Problem 163. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing, continuous probability density function and let $m_p = \int_0^\infty x^p f(x)dx$ be its pth moment. Show that $((p+1)m_p)^{\frac{1}{p+1}}$ is increasing in $p \in [0, \infty)$.
[*Hint:* Consider a measure ν such that $\nu[x, \infty) = f(x)$ and relate m_p to ν.]

Problem 164 (Existence of Markov chains). Let S be a countable set (with the power set sigma algebra). Two ingredients are given: *A transition matrix*, that is, a function $p : S \times S \to [0, 1]$ be a function such that $p(x, \cdot)$ is a probability mass function on S for each $x \in S$. (1) *An initial distribution*, that is a probability mass function μ_0 on S.

For $n \geq 0$ define the probability measure ν_n on S^{n+1} (with the product sigma algebra) by

$$\nu_n(A_0 \times A_1 \times \ldots \times A_n) = \sum_{(x_0,\ldots,x_n) \in A_0 \times \ldots \times A_n} \mu_0(x_0) \prod_{j=0}^{n-1} p(x_j, x_{j+1}).$$

Show that ν_n form a consistent family of probability distributions and conclude that a Markov chain with initial distribution μ_0 and transition matrix p exists.

Problem 165. Show that it is not possible to define uncountably many independent Ber(1/2) random variables on the probability space $([0, 1], \mathcal{B}, \lambda)$.

Problem 166. Let $(\Omega_i, \mathcal{F}_i, \mathbf{P}_i)$, $i \in I$, be probability spaces and let $\Omega = \times_i \Omega_i$ with $\mathcal{F} = \otimes_i \mathcal{F}_i$ and $\mathbf{P} = \otimes_i \mathbf{P}_i$. If $A \in \mathcal{F}$, show that for any $\varepsilon > 0$, there is a cylinder set B such that $\mathbf{P}(A \Delta B) < \varepsilon$.

Problem 167. Let $A_1, A_2, \ldots$ be a sequence of events in $(\Omega, \mathcal{F}, \mathcal{P})$. Let p_k be the probability that at least one of the events $A_k, A_{k+1}, \ldots$ occurs.

(1) If $\inf_k p_k > 0$, then show that A_n occurs infinitely often, w.p.1.

(2) If $p_k \to 0$, then show that only finitely many A_n occur, w.p.1.

Problem 168. Let A_n be events in a probability space such that $\sum_n \mathbf{P}(A_n \Delta B_n) < \infty$ for some sequence of independent events B_n. Show that $\mathbf{P}(A_n \text{ i.o.})$ is 0 if $\sum_n P(A_n) < \infty$ and 1 if $\sum_n \mathbf{P}(A_n) = \infty$.

Problem 169. Show that there is no function $\varphi : [0,1] \to [0,1]$ such that $\varphi(x) \to 0$ as $x \to 0$ for which $\sum_n \varphi(\mathbf{P}(A_n)) = \infty$ implies $\mathbf{P}(A_n \text{ i.o.}) > 0$. [*Remark*: This is to emphasize that the second Borel-Cantelli cannot be got some independence assumption. For example, even if $\sum_n \mathbf{P}(A_n)^{10} = \infty$, we may still have $\mathbf{P}(A_n \text{ i.o.}) = 0$.]

Problem 170. Let A_n be events in a probability space.

(1) If there exist $1 = N_1 < N_2 < \ldots \to \infty$ such that if $B_k = A_{N_k} \cup \ldots \cup A_{N_{k+1}-1}$ satisfy $\sum_k \mathbf{P}(B_k) < \infty$, then $\mathbf{P}(A_n \text{ i.o.}) = 0$.

(2) Is the converse true? If $\mathbf{P}(A_n \text{ i.o.}) = 0$, must there exist (N_k) such that $\sum_k \mathbf{P}(B_k) < \infty$?

Problem 171. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that $\limsup_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1$ *a.s.*

(2) Let c_n be (non-random) complex numbers. Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} c_n \xi_n z^n$ is almost surely equal to the radius of convergence of the non-random power series $\sum_{n=0}^{\infty} c_n z^n$.

Problem 172. Let $(X_n)_n$ be a sequence of random variables such that $\{X_{2n} : n \geq 1\}$ are independent and $\{X_{2n-1} : n \geq 1\}$ are independent. Does it follow that the tail sigma algebra of the sequence $(X_n)_n$ is trivial?

Problem 173. Let X_n be independent random variables with $X_n \sim \text{Ber}(p_n)$. For $k \geq 1$, find a sequence (p_n) so that almost surely, the sequence $X_1, X_2, \ldots$ has infinitely many segments of ones of length k but only finitely many segments of ones of length $k+1$. By a segment of length k we mean a consecutive sequence $X_i, X_{i+1}, \ldots, X_{i+k-1}$.

Problem 174. Let ℓ_n be numbers in $[0, 2\pi)$. Let $\theta_1, \theta_2, \ldots$ be i.i.d. Unif$[0, 2\pi]$ and let J_k be the arc of the unit circle $S^1 = \{e^{it} : 0 \leq t < 2\pi\}$ with center $e^{i\theta_k}$ and having length ℓ_k. Let $J = \cup_k J_k$. Show that the following are equivalent:

(1) $S^1 \setminus J$ has zero Lebesgue measure in S^1, a.s.

(2) $\sum_n \ell_n = \infty$.

[*Hint:* First fix $x \in S^1$ and consider the event that x is covered by infinitely many J_n.]

Problem 175. (Ergodicity of product measure). This problem guides you to a proof of a different zero-one law.

(1) Consider the product measure space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \otimes_{\mathbb{Z}}\mu)$ where $\mu \in \mathcal{P}(\mathbb{R})$. Define $\tau : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(\tau\omega)_n = \omega_{n+1}$. Let $\mathcal{I} = \{A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}}) : \tau(A) = A\}$. Then, show that $\mathcal{I}$ is a sigma-algebra (called the invariant sigma algebra) and that every event in $\mathcal{I}$ has probability equal to 0 or 1.

(2) Let $X_n, n \geq 1$ be i.i.d. random variables on a common probability space. Suppose $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a measurable function such that $f(x_1, x_2, \ldots) = f(x_2, x_3, \ldots)$ for any $(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. Then deduce from the first part that the random variable $f(X_1, X_2, \ldots)$ is a constant, *a.s.*

[**Hint:** Approximate A by cylinder sets. Use translation by τ^m to show that $\mathbf{P}(A) = \mathbf{P}(A)^2$.]

Problem 176. Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Show that there exist $\varepsilon, \varepsilon' \in \{-1, +1\}^n$ such that

(1) $\|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n\| \leq \sqrt{n}$,

(2) $\|\varepsilon'_1 v_1 + \ldots + \varepsilon'_n v_n\| \geq \sqrt{n}$.

[*Hint:* Probabilistic method]

Problem 177. If X and Y are i.i.d. random variables, show that the (closed) support of the distribution of $X - Y$ contains 0.

Problem 178. If $X \geq 0$ and $\mathbf{E}[X] = m$, then show that $\mathbf{P}\{X \leq m\} > 0$. Is there is an absolute lower bound (meaning, the bound does not depend on X) for $\mathbf{P}\{X \leq m\}$?

Problem 179. Assume $\sigma^2 := \mathrm{Var}(X) < \infty$ and $\mathbf{E}[X] = 0$. Show that $\mathbf{P}\{X \geq t\} \leq \frac{\sigma^2}{\sigma^2 + t^2}$ for $t > 0$. [*Hint:* Consider $(X - t)_{\pm}$.].

[*Note:* Compare with direct application of Chebyshev's inequality.]

Problem 180. If $X \geq 0$ has finite second moment, show that $\mathbf{P}\{X = 0\} \leq \frac{\mathrm{Var}(X)}{\mathbf{E}[X^2]}$.

Problem 181. Let X be a random variable with mean 0. Assume that $\tau = \|X\|_4$ and let $\sigma = \|X\|_2$ are finite. Let $\gamma = \tau/\sigma$. Show that

$$\mathbf{P}\{|X| \geq k\sigma\} \leq \begin{cases} \frac{1}{k^2} & \text{for any } k \geq 1, \\ \frac{\gamma^4 - 1}{\gamma^4 + k^4 - 2k^2} & \text{if } k \geq \gamma^2. \end{cases}$$

[*Remark:* Strengthening of Chebyshev for high deviations, assuming 4th moment.]

Problem 182. (Chung-Erdös inequality).

(1) Let A_i be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n}\mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n}\mathbf{P}(A_k \cap A_\ell)}$$

(2) Place r_m balls in m bins at random and count the number of empty bins Z_m. Fix $\delta > 0$. If $r_m > (1+\delta)m\log m$, show that $\mathbf{P}(Z_m > 0) \to 0$ while if $r_m < (1-\delta)m\log m$, show that $\mathbf{P}(Z_m > 0) \to 1$.

Problem 183. Let $\mu \in \mathcal{P}(\mathbb{R})$ and assume that it has finite mean M and variance σ^2. If $M \in [a,b]$ and $\mu(a,b) = 0$, then show that $\sigma^2 \geq (M-a)(b-M)$.

Problem 184. Give example of an infinite sequence of pairwise independent random variables for which Kolmogorov's zero-one law fails.

Problem 185. Let $X_1,\ldots,X_n$ be random variables with $\mathbf{E}[X_i] = 0$ and $|X_i| \leq B_i$ a.s. Assume that $\mathbf{E}[X_{i_1}\ldots X_{i_k}] = 0$ for any $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Check that the proof of Hoeffding's inequality goes through for $S_n = X_1 + \ldots + X_n$.

Problem 186. Let $X_1,\ldots,X_n$ be independent random variables with $\mathbf{E}[X_k] = 0$ and $\mathbf{E}[X_k^2] = \sigma_k^2$. Let $S_k = X_1 + \ldots + X_k$ (so $S_0 = 0$) and let $S_n^* = \max\limits_{0\leq k\leq n} |S_k|$. Show that $\mathbf{E}[S_n^*] \leq 2\tau_n$ where $\tau_n^2 = \sigma_1^2 + \ldots + \sigma_n^2$.

Problem 187. Suppose X_n are independent random variables and $\sum\limits_n X_n$ converges a.s. Show that $\sum\limits_n \mathbf{P}\{|X_n| > A\} < \infty$ for any $A > 0$.

Problem 188. Let $X_n \sim \text{Unif}[-a_n, a_n]$ be independent. If $\sum\limits_n a_n^2$ converges, show that $\prod_{n=1}^{\infty}(1+X_n)$ converges a.s.

Problem 189. Let $F(s) = \prod_p (1 - X_p p^{-s})^{-1}$ where the product is over all prime numbers and X_p are i.i.d. $\text{Ber}_{\pm}(1/2)$ random variables. Show that $F(s)$ converges a.s. for $s > \frac{1}{2}$. [*Remark*: Compare this with the famous discovery of Euler that $\prod_p (1-p^{-s})^{-1} = \sum_{n\geq 1} n^{-s}$, which converges for $s > 1$]

Problem 190. Let X_i, $i \in I$ be random variables on a probability space. Suppose that for some $p > 0$ and $M < \infty$ we have $\mathbf{E}[|X_i|^p] \leq M$ for all $i \in I$. Show that the family $\{X_i : i \in I\}$ is tight (by which we mean that $\{\mu_{X_i} : i \in I\}$ is tight, where μ_{X_i} is the distribution of X_i).

Problem 191. Let X_i be i.i.d. random variables with zero mean and finite variance. Let $S_n = X_1 + \ldots + X_n$. Show that the collection $\{\frac{1}{\sqrt{n}}S_n : n \geq 1\}$ is tight. [**Note:** Tightness is essential for convergence in distribution. In the case at hand, convergence in distribution to $N(0,1)$ is what is called central limit theorem. We shall see it later.]

Problem 192. Suppose each of $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\left| \frac{N_n}{n} - e^{-\lambda} \right| > \delta \right) \to 0$$

Problem 193. Let $\xi_1, \ldots, \xi_n$ be i.i.d. tosses of a p-coin. If $\xi_{k+1} = \xi_{k+1} = \ldots = \xi_{k+m} = 1$ but $\xi_k = \xi_{k+m+1} = 0$, we say that $(k, \ldots, k+m+1)$ is a run of heads of length exactly equal to m. Let $T_{n,m}$ denote the number of runs of length exactly equal to m.

(1) For fixed m, show that $\frac{T_{n,m}}{n} \stackrel{P}{\to} q^2 p^m$ as $n \to \infty$.

(2) Does your proof work for $m = m_n$ increasing with n? If so how fast can it grow?

Problem 194. A random graph $\mathcal{G}_n$ with vertex set $[n] = \{1, \ldots, n\}$ is built by connecting every pair of distinct vertices with probability p_n. Show that for any $\varepsilon > 0$,

$$\mathbf{P}\{\mathcal{G}_n \text{ has an isolated vertex}\} \to \begin{cases} 1 & \text{if } p_n < (1-\varepsilon)\frac{\log n}{n} \\ 0 & \text{if } p_n > (1+\varepsilon)\frac{\log n}{n}. \end{cases}$$

[*Hint:* Consider the number of isolated vertices.]

Problem 195. A box contains n distinct pairs of gloves. Two gloves are drawn at random and then returned to the box. Repeat this till each *pair* of gloves has been drawn at least once. If T_n is the number of draws, then can you find a deterministic sequence a_n such that $\mathbf{P}\{1-\varepsilon \leq \frac{T_n}{a_n} \leq 1+\varepsilon\} \to 1$ as $n \to \infty$, for any $\varepsilon > 0$? (in language to be introduced later, we write this as $\frac{T_n}{a_n} \stackrel{P}{\to} 1$).

Problem 196. Repeat the previous problem for a box containing $2n$ hats. Again hats are drawn, two at a time, till every pair is seen. (The difference is that hats don't come in natural pairs).

Problem 197. Let $G_{n,p}$ be the random graph with vertex set $[n]$ and edge set $\{\{i,j\} : X_{i,j} = 1\}$, where $X_{i,j}$, $i < j$, are i.i.d. Ber(p). Let R be the size of the largest clique in $G_{n,p}$ (a clique is a subset S of vertices such that every pair of vertices in S is connected by an edge). Show that $\mathbf{P}\{a \log n \leq R_n \leq b \log n\} \to 1$ as $n \to \infty$, for some $0 < a < b < \infty$.

Problem 198. Consider a Galton-Watson tree $\mathcal{T}$ with offspring variable L with $\mathbf{P}\{L = k\} = p_k$, $k \geq 0$. Let $L_1, L_2, \ldots$ be i.i.d. copies of L and let τ be the first time that the random walk $S_n = \sum_{j=1}^{n}(L_j - 1)$ hits the level -1. Show that for any $k \geq 1$,

$$\mathbf{P}\{|\mathcal{T}| = k\} = \mathbf{P}\{\tau = k\}.$$

Problem 199. Consider Galton-Watson tree with offspring distribution Pois(λ). For $\lambda < 1$, show that $\mathbf{P}\{|\mathcal{T}| \geq k\} \leq e^{-ck}$ for all k, for some $c > 0$ (that may depend on λ). [*Remark:* Not only is $\mathcal{T}$ finite, it is highly unlikely to be large.]

Problem 200. Let $A_1, A_2, \ldots$ be i.i.d. uniform random subsets of $[n]$ (i.e., $\mathbf{P}(A_1 = S) = 2^{-n}$ for each $S \subseteq [n]$). Imagine sampling $A_1, A_2, \ldots$ suvvessively and let T_n be the first time when we have two subsets that are disjoint from each other. Show that $T_n \approx (2/\sqrt{3})^n$ in the sense that

$$\mathbf{P}\left\{T_n \geq \left(\frac{2}{\sqrt{3}}\right)^n h_n\right\} \to \begin{cases} 0 & \text{if } h_n \to \infty, \\ 1 & \text{if } h_n \to 0. \end{cases}$$

Problem 201. Same setting as the previous problem, but now let T_n be the first time some subset contains another. Analyse T_n as in that problem.

Problem 202. Let X_n be i.i.d random variables such that $\mathbf{E}[|X_1|] < \infty$. Define the random power series $f(z) = \sum_{k=0}^{\infty} X_n z^n$. Show that almost surely, the radius of convergence of f is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_n z^n$ is given by $(\limsup |c_n|^{\frac{1}{n}})^{-1}$].

Problem 203. Let $X_1, X_2, \ldots$ be i.i.d. fair coin tosses. Let L_n be the length of the longest run of heads in $X_1, \ldots, X_n$ (a run is a segment of consecutive tosses). Show that for any $\varepsilon > 0$,

$$\mathbf{P}\{(1-\varepsilon)\log_2 n \leq L_n \leq (1+\varepsilon)\log_2 n\} \to 1.$$

Problem 204. Let $X_n \sim \text{Ber}(n^{-\alpha})$ be independent, $\alpha > 0$. What is the largest k for which the sequence $X_1, X_2, X_3, \ldots$ contains a sequence of k ones, almost surely?

Problem 205. How does the analysis in the coupon collector problem change if one waits till each coupon is seen at least two times?

Problem 206. Let F, G be CDFs such that $F(t) \le G(t)$ for all $t \in \mathbb{R}$ (we say that F *stochastically dominates* G). Show that there exists a probability space and random variables X, Y on it such that $X \sim F$, $Y \sim G$ and $X \ge Y$ a.s.

Problem 207. Let X_n, Y_n be random variables on a common probability space such that $\mathbf{P}\{|X_n - Y_n| \ge \frac{1}{n^2}\} \le \frac{1}{n^2}$. If $\sum\limits_n X_n$ converges a.s., show that $\sum\limits_n Y_n$ converges a.s.

Problem 208. Let $X_n \sim \text{Ber}(\frac{1}{2})$ be i.i.d. and let $Y_n \sim \text{Ber}(\frac{1}{2} + \frac{1}{2n^2})$ be independent. Let $X = (X_1, X_2, \ldots)$ and $Y = (Y_1, Y_2, \ldots)$. If A is a Borel subset of $\mathbb{R}^{\mathbb{N}}$, then show that $\mathbf{P}\{X \in A\} > 0$ if and only if $\mathbf{P}\{Y \in A\} > 0$. [*Hint:* Use coupling]

Problem 209. (1) Let X be a real values random variable with finite variance. Show that $f(a) := \mathbf{E}[(X-a)^2]$ is minimized at $a = \mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a) = \mathbf{E}[|X - a|]$? [**Hint:** First consider X that takes finitely many values with equal probability each].

Problem 210. Let X_i be i.i.d. Cauchy random variables with density $\frac{1}{\pi(1+t^2)}$. Show that $\frac{1}{n}S_n$ fails the weak law of large numbers by completing the following steps.

(1) Show that $t\mathbf{P}\{|X_1| > t\} \to c$ for some constant c.

(2) Show that if $\delta > 0$ is small enough, then $\mathbf{P}\{|\frac{1}{n-1}S_{n-1}| \ge \delta\} + \mathbf{P}\{|\frac{1}{n}S_n| \ge \delta\}$ does not go to 0 as $n \to \infty$ [*Hint:* Consider the possibility that $|X_n| > 2\delta n$].

(3) Conclude that $\frac{1}{n}S_n$ does not converge t0 0 in probability. [*Extra:* With a little more effort, you can try showing that there does not exist deterministic numbers a_n such that $\frac{1}{n}S_n - a_n \stackrel{P}{\to} 0$].

Problem 211. Let X_n, X be random variables on a common probability space.

(1) If $X_n \stackrel{P}{\to} X$, show that some subsequence $X_{n_k} \stackrel{a.s.}{\to} X$.

(2) If every subsequence of X_n has a further subsequence that converges almost surely to X, show that $X_n \stackrel{P}{\to} X$.

Problem 212. For $\mathbb{R}^d$-valued random vectors X_n, X, the notions of convergence almost surely, in probability and in distribution are well-defined. If $X_n = (X_{n,1}, \ldots, X_{n,d})$ and $X = (X_1, \ldots, X_d)$, which of the following is true? Justify or give counterexamples.

(1) $X_n \stackrel{a.s.}{\to} X$ if and only if $X_{n,k} \stackrel{a.s.}{\to} X_k$ for $1 \le k \le d$.

(2) $X_n \xrightarrow{P} X$ if and only if $X_{n,k} \xrightarrow{P} X_k$ for $1 \le k \le d$.

(3) $X_n \xrightarrow{d} X$ if and only if $X_{n,k} \xrightarrow{d} X_k$ for $1 \le k \le d$.

Problem 213. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ (all r.v.s on the same probability space), show that $aX_n + bY_n \xrightarrow{P} aX + bY$ and $X_nY_n \xrightarrow{P} XY$. [**Hint:** You could try showing more generally that $f(X_n, Y_n) \to f(X, Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$.]

Problem 214. Let X_n, Y_n, X, Y be random variables on a common probability space.

(1) Suppose that X_n is independent of Y_n for each n (no assumptions about independence across n). If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent. Further, $aX_n + bY_n \xrightarrow{d} aU + bV$.

(2) Give counterexample to show that the previous statement is false if the assumption of independence of X_n and Y_n is dropped.

Problem 215. If X_n are independent random variables and $X_n \xrightarrow{P} X$. Show that X is a constant random variable.

Problem 216. If X_n, Y_n are independent for each n and $X_n + Y_n \xrightarrow{P} 0$. Show that there are numbers y_n such that $X_n + y_n \xrightarrow{P} 0$.

Problem 217. Let $a_n, a \in \mathbb{R}$ and $a_n \to a$. Let $\mu_n = \frac{1}{n}(\delta_{a_1} + \ldots + \delta_{a_n})$ be the probability measure that puts mass $\frac{1}{n}$ at each a_k, $k \le n$ (with appropriate multiplicity). Show that μ_n converges in distribution and find the limit.

Problem 218. Let $\mu_n = \frac{1}{n-1}\sum_{k=1}^{n-1} \delta_{f(\frac{k}{n})}$, where $f : (0,1) \to \mathbb{R}$ is some continuous function. Show that μ_n converges in distribution and describe the limit. Find the limit explicitly when $f(x) = x^p$.

Problem 219. Suppose $\mu_n \xrightarrow{d} \mu$. Let $c_{n,k} \ge 0$ for $1 \le k \le n$ such that $c_{n,1} + \ldots + c_{n,n} = 1$ for each n and such that $c_{n,j} \to 0$ as $n \to \infty$ for each j. Let $\nu_n = c_{n,1}\mu_1 + \ldots + c_{n,n}\mu_n$. Show that $\nu_n \xrightarrow{d} \mu$.

Problem 220. Suppose $X_n \xrightarrow{P} X$. Let $c_{n,k} \ge 0$ for $1 \le k \le n$ such that $c_{n,1} + \ldots + c_{n,n} = 1$ for each n and such that $c_{n,j} \to 0$ as $n \to \infty$ for each j. Let $Y_n = c_{n,1}X_1 + \ldots + c_{n,n}X_n$. Show that $Y_n \xrightarrow{P} X$.

Problem 221. Suppose $X_n \stackrel{P}{\to} X$. Use the previous exercise to deduce the following.

(1) The Cesàro sums $\frac{1}{n}(X_1 + \ldots + X_n) \stackrel{P}{\to} X$ as $n \to \infty$.

(2) The Abelian sums $\frac{1}{1-r} \sum\limits_{k=0}^{n} r^k X_k \stackrel{P}{\to} X$ as $r \uparrow 1$ (if you prefer, take a sequence $r_n \uparrow 1$).

Problem 222. For $\mathbb{R}^d$-valued random vectors X_n, X, we say that $X_n \stackrel{P}{\to} X$ if $\mathbf{P}(\|X_n - X\| > \delta) \to 0$ for any $\delta > 0$ (here you may take $\|\cdot\|$ to denote the usual norm, but any norm on $\mathbb{R}^d$ gives the same definition).

(1) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, show that $(X_n, Y_n) \stackrel{P}{\to} (X, Y)$.

(2) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, show that $X_n + Y_n \stackrel{P}{\to} X + Y$ and $\langle X_n, Y_n \rangle \stackrel{P}{\to} XY$. [**Hint:** Show more generally that $f(X_n, Y_n) \stackrel{P}{\to} f(X, Y)$ for any continuous function f by using the previous problem for random vectors].

Problem 223. (1) If X_n, Y_n are independent random variables on the same probability space and $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{d}{\to} Y$, then $(X_n, Y_n) \stackrel{d}{\to} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and U, V are independent.

(2) If $X_n \stackrel{d}{\to} X$ and $Y_n - X_n \stackrel{P}{\to} 0$, then show that $Y_n \stackrel{d}{\to} X$.

Problem 224. Let $Y_n = \frac{|X_n|}{1+|X_n|}$. Show that $X_n \stackrel{P}{\to} 0$ if and only if $Y_n \stackrel{L^1}{\to} 0$.

Problem 225. Let X_n be a sequence of random variables on a common probability space. Show that there exists a (non-random) sequence of real numbers a_n such that $a_n X_n \stackrel{a.s.}{\to} 0$.

Problem 226. Show that the the following are equivalent conditions for tightness of a sequence $\{X_n\}$.

(1) $c_n X_n \stackrel{P}{\to} 0$ whenever $c_n \to 0$.

(2) $\mathbf{P}\{|X_n| > M_n\} \to 0$ whenever $M_n \to \infty$.

Problem 227. Show that the the following are equivalent conditions for uniform integrability of a sequence $\{X_n\}$.

(1) $c_n X_n \stackrel{L^1}{\to} 0$ whenever $c_n \to 0$.

(2) $\mathbf{E}[|X_n| \mathbf{1}_{|X_n| > M_n}] \to 0$ whenever $M_n \to \infty$.

Problem 228. Two common ways to check uniform integrability of a family of random variables are (1) Domination by an integrable random variable, (2) L^p-boundedness for some $p > 1$. Show that neither condition implies the other.

Problem 229. Let $X_i \sim \mu$ be i.i.d. If S_n converge a.s., show that $\mu = \delta_0$.

Problem 230. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Show that $\int f d\mu_n \to \int f d\mu$ if the sequence $\{f d\mu_n\}$ is tight (i.e., given $\varepsilon > 0$, there is some $M < \infty$ such that $\int_{[-M,M]^c} |f| d\mu_n \leq \varepsilon$ for all n).

Problem 231. Let α_n be a sequence of real numbers. Assume that for each n, there is a $\mu_n \in \mathcal{P}(\mathbb{R})$ such that $\int x^k d\mu_n(x) = \alpha_k$ for $1 \leq k \leq n$. Show that there is a $\mu \in \mathcal{P}(\mathbb{R})$ such that $\int x^k d\mu(x) = \alpha_k$ for all $k \geq 1$.

Problem 232. For each mode of convergence (almost sure, in probability, in distribution, in L^p), decide whether the following statement is true: "If $X_n \to X$ then $\frac{1}{n} S_n \to X$", where $S_n = X_1 + \ldots + X_n$.

[*Remark:* The question is motivated by the analogous fact for convergence of numbers.]

Problem 233. Let $X_1, X_2, \ldots$ be i.i.d from μ. For each n, define the random probability measure $\mu_n = \frac{1}{n}(\delta_{X_1} + \ldots + \delta_{X_n})$. If F_n, F are the cumulative distribution functions of μ_n and μ, show that for any $x \in \mathbb{R}$, we have $F_n(x) \overset{a.s.}{\to} F(x)$.

Problem 234. Let X_n be independent with $X_n \sim \text{Poisson}(\lambda_n)$. Let $S_n = X_1 + \ldots + X_n$.

(1) If $\lambda_n = 1 + \frac{1}{n^b}$ for some $b > 0$, show that $\frac{S_n}{n} \overset{P}{\to} 1$.

(2) If $\lambda_n = 1 + \frac{1}{n^b}$ for some $b > 1$, show that $\frac{S_n}{n} \overset{a.s.}{\to} 1$.

Problem 235. Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite expectation m. Show that

$$\mathbf{E}\left[\left| \frac{S_n}{n} - m \right| \right] \to 0.$$

Problem 236. Let $X_1, X_2, \ldots$ be i.i.d. random variables. Let G_n be the geometric mean of $X_1, \ldots, X_n$. In each of the following cases, show that G_n converges almost surely to a constant and find the constant. (a) $X_1 \sim \text{Unif}[0, 1]$, (b) $X_1 \sim \text{Exp}(1)$.

Problem 237. Suppose X_n are i.i.d with $\mathbf{E}[|X_1|^4] < \infty$. Show that there is some constant C (depending on the distribution of X_1) such that $\mathbf{P}\left(|n^{-1}S_n - \mathbf{E}[X_1]| > \delta\right) \le Cn^{-2}$. (What is your guess if we assume $\mathbf{E}[|X_1|^6] < \infty$? You don't need to show this in the homework).

Problem 238. (1) (**Skorokhod's representation theorem**) If $X_n \stackrel{d}{\to} X$, then show that there is a probability space with random variables Y_n, Y such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \stackrel{a.s.}{\to} Y$. [**Hint:** Try to construct Y_n, Y on the canonical probability space $([0,1], \mathcal{B}, \mu)$]

(2) If $X_n \stackrel{d}{\to} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \stackrel{d}{\to} f(X)$. [**Hint:** Use the first part]

Problem 239. Suppose X_i are i.i.d with the Cauchy distribution (density $\pi^{-1}(1+x^2)^{-1}$ on $\mathbb{R}$). Note that X_1 is not integrable. Then, show that $\frac{S_n}{n}$ does not converge in probability to any constant. [**Hint:** Try to find the probability $\mathbf{P}(X_1 > t)$, and then use it].

Problem 240. Let $X_1, X_2, \ldots$ be i.i.d. random variables with symmetric Pareto distribution with density $\frac{1}{2x^2}$ for $|x| > 1$.

(1) Show that $|X_n| \ge n$ for infinitely many n, almost surely.

(2) Deduce that $\frac{S_n}{n}$ does not converge, a.s. Why does this not contradict SLLN?

Problem 241. Let X_n be i.i.d. positive random variables and let $M_n = \max\{X_1, \ldots, X_n\}$.

(1) If $\mathbf{E}[X_1] < \infty$, show that $\frac{M_n}{S_n} \stackrel{P}{\to} 0$.

(2) Give an example of a distribution with $\mathbf{E}[X_1] = \infty$ for which $\frac{M_n}{S_n}$ does not converge to 0 in probability.

(3) Is there any distribution with $\mathbf{E}[X_1] = \infty$ for which we do have $\frac{M_n}{S_n} \stackrel{P}{\to} 0$?

[*Remark:* When this fails, it means that one of $X_1, \ldots, X_N$ is as large as their sum. With light tailed random variables, no single term contributes too much to the total]

Problem 242. Let $U \sim \text{Uniform}[0,1]$ and $X_n = \sin(nU)$. Show that X_n converges in distribution and find the limit.

Problem 243. Let $X_1, X_2, \ldots$ be i.i.d. Unif[0, 1] random variables and let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denotes the *order statistics* (i.e., $X_{(k)}$ is the kth smallest among X_is, e.g., $X_{(1)} = \min X_i$).

(1) Show that $nX_{(1)} \stackrel{d}{\to} \text{Exp}(1)$ as $n \to \infty$.

(2) For any fixed $k \geq 1$ show that $(nX_{(1)}, n(X_{(2)} - X_{(1)}), \dots, n(X_{(k)} - X_{(k-1)})) \stackrel{d}{\to} (\xi_1, \dots, \xi_n)$, where ξ_j are i.i.d. Exp(1) random variables. [*Remark:* This required convergence in distribution in higher dimensions. If not clear what that is, omit the problem]

Problem 244. Repeat the previous problem if X_i are i.i.d. from a probability density f on $\mathbb{R}_+$ with $f(0) > 0$ (the exponentials will change parameters).

Problem 245. Let X_i are i.i.d. from density px^{p-1} on $[0, 1]$, for some $p > 0$. Find an appropriate limiting law for $X_{(1)}$ as in the previous problem (note that if $p \neq 1$, the density vanishes at 0 or is infinite).

Problem 246. Let $X_1, X_2, \dots$ be i.i.d. standard Cauchy random variables and let $M_n = \max\{X_1, \dots, X_n\}$. Show that $\frac{n}{M_n} \stackrel{d}{\to} \text{Exp}(1)$.

Problem 247. Suppose $0 \leq X_1 \leq X_2 \leq \dots$. Assume that $\frac{\mathbf{E}[X_n]}{n^\alpha} \to A$ and $\text{Var}(X_n) \leq Bn^{2\beta}$ for some $0 < A, B < \infty$ and $0 < \beta < \alpha < \infty$. Show that $\frac{X_n}{n} \stackrel{a.s.}{\to} A$.

Problem 248. Let $G_1, G_2, \dots$ be i.i.d Geometric(p) random variables (this means $\mathbf{P}(G_1 = k) = p(1-p)^{k-1}$ for $k \geq 1$). Let $X_1, X_2, \dots$ be i.i.d random variables with $\mathbf{E}[|X_1|] < \infty$. Define $N_k := G_1 + G_2 + \dots + G_k$. Show that as $k \to \infty$,

$$\frac{X_1 + X_2 + \dots + X_{N_k}}{k} \stackrel{P}{\to} \frac{1}{p}\mathbf{E}[X_1]$$

Problem 249. Let $X_1, X_2, \dots, X_n$ be i.i.d. points sampled uniformly from the unit disk $\mathbb{D} \subseteq \mathbb{R}^2$. Let $R_n = \min_{1 \leq k \leq n} \|X_k\|$.

(1) Show that $\sqrt{n}R_n \stackrel{d}{\to} R$ where R has the Rayleigh density xe^{-x^2} on $\mathbb{R}_+$.

(2) How do things change if X_i are uniform on the unit ball in $\mathbb{R}^3$?

Problem 250. Show that for any $p \geq 1$,

$$\lim_{n \to \infty} \int_{[0,1]^n} \frac{x_1^p + \dots + x_n^p}{x_1 + \dots + x_n} dx_1 \dots dx_n = \frac{2}{p+1}.$$

[*Hint:* Do it without having to flex your muscles too much. Use probability!]

Problem 251. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 1-periodic ($f(t+1) = f(t)$ for all t). Show that for a.e. $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\overline{kx}) \to \int_0^1 f(t)dt.$$

where $\bar{y} = y \pmod 1$.

Problem 252. Let $\Omega_n = \{x = (x_1, \ldots, x_n) : 0 \le x_i \in \mathbb{Z},\ x_1 + \ldots + x_n = k_n\}$, a finite subset of $\mathbb{Z}^n$. Assume that $\frac{k_n}{n} \to \alpha > 0$. Let $X_n = (X_{n,1}, \ldots, X_{n,n})$ be uniformly randomly sampled from Ω_n.

(1) Show that $X_{n,1} \stackrel{d}{\to} \text{Geo}(p)$ for some p and find p in terms of α.

(2) For any $k \ge 1$, show that $(X_{n,1}, \ldots, X_{n,k}) \stackrel{d}{\to} (\xi_1, \ldots, \xi_k)$, where ξ_j are i.i.d. $\text{Geo}(p)$ random variables.

Problem 253. Fix $\alpha > 0$ and let $\Omega_n = \{x = (x_1, \ldots, x_n) : x_i > 0,\ x_1 + \ldots + x_n = \alpha n\}$, a bounded open set in $\mathbb{R}^n$. Let $X_n = (X_{n,1}, \ldots, X_{n,n}) \sim \text{Unif}(\Omega_n)$ (normalized Lebesgue measure).

(1) Show that $X_{n,1} \stackrel{d}{\to} \text{Exp}(1/\alpha)$.

(2) For any $k \ge 1$, show that $(X_{n,1}, \ldots, X_{n,k}) \stackrel{d}{\to} (\xi_1, \ldots, \xi_k)$, where ξ_j are i.i.d. $\text{Exp}(1/\alpha)$ random variables.

Problem 254. Let $\{X_i\}_{i\in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) If $\{X_i\}_{i\in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

(2) Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(x) \to \infty$ as $x \to \infty$. If $\sup_i \mathbf{E}[|X_i| h(|X_i|)] < \infty$, show that $\{X_i\}_{i\in I}$ is uniformly integrable. In particular, if $\sup_i \mathbf{E}[|X_i|^p] < \infty$ for some $p > 1$, then $\{X_i\}$ is uniformly integrable.

Problem 255. Let X_n be a sequence of random variables with zero means, unit variances. Assume that $|\text{Cov}(X_n, X_m)| \le \delta(|n-m|)$ where $\delta(k) \to 0$ as $k \to \infty$. Show that $\frac{1}{n}S_n \stackrel{P}{\to} 0$.

How to modify the conclusion if we change the unit variance assumption to "$\mathbf{E}[X_n^2] = \sigma_n^2$"?

Problem 256. Let X_n be i.i.d with $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = \frac{1}{2}$. Show that for any $\gamma > \frac{1}{2}$,

$$\frac{S_n}{n^\gamma} \stackrel{a.s.}{\to} 0.$$

[**Remark:** Try to imitate the proof of SLLN under fourth moment assumption. If you write the proof correctly, it should go for any random variable which has moments of all orders. You do not need to show this for the homework].

Problem 257. Let X_k be independent random variables with $\mathbf{P}\{X_k = k^a\} = \mathbf{P}\{X_k = -k^a\} = \frac{1}{2}$ for some $a \geq 0$. Show that $\frac{S_n}{n} \stackrel{P}{\to} 0$ if and only if $a < \frac{1}{2}$.

Problem 258. Let A_n be events in a common probability space such that $\mathbf{P}(A_n) \geq p$ for all n. Show that $\mathbf{P}\{A_n \text{ i.o.}\} \geq p$.

Problem 259. Let A_n be events in a common probability space. Assume that for some $n_1 < n_2 < \ldots$ we have $\mathbf{P}\left(\bigcup_{n=n_k+1}^{n_{k+1}} A_n\right) \geq p$ for all k. Show that $\mathbf{P}\{A_n \text{ i.o.}\} \geq p$.

Problem 260. Let X_n be independent real-valued random variables.

(1) Show by example that the event $\{\sum X_n$ converges to a number in [1,3]$\}$ can have probability strictly between 0 and 1.

(2) Show that the event $\{\sum X_n$ converges to a finite number$\}$ has probability zero or one.

Problem 261. Let X_n be i.i.d exponential(1) random variables.

(1) If b_n is a sequence of numbers that converge to 0, show that $\limsup b_n X_n$ is a constant (a.s.). Find a sequence b_n so that $\limsup\ b_n X_n = 1$ a.s.

(2) Let M_n be the maximum of $X_1, \ldots, X_n$. If $a_n \to \infty$, show that $\limsup \frac{M_n}{a_n}$ is a constant (a.s.). Find a_n so that $\limsup \frac{M_n}{a_n} = 1$ (a.s.).

[**Remark:** Can you do the same if X_n are i.i.d N(0,1)? Need not show this for the homework, but note that the main ingredient is to find a simple expression for $\mathbf{P}(X_1 > t)$ asymptotically as $t \to \infty$].

Problem 262. Let X_k be non-degenerate i.i.d. random variables with $\mathbf{E}|X_1|^\delta < \infty$ for some $\delta > 0$.

(1) Show that $\frac{X_n}{S_n} \stackrel{P}{\to} 0$.

(2) Give counterexample to show that $\frac{X_n}{S_n}$ need not converge to 0 a.s.

Problem 263. Let X_n be i.i.d real valued random variables with common distribution μ. For each n, define the random probabilty measure μ_n as $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{X_k}$. Let F_n be the CDF of μ_n. Show that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \stackrel{a.s.}{\to} 0 \quad a.s.$$

Problem 264. Let $X_1, X_2, \dots$ be i.i.d. with distribution $\mu \in \mathcal{P}(\mathbb{R})$. Recall that the support of μ is the smallest closed set K with $\mu(K) = 1$. Show that $\overline{\{X_1, X_2, \dots\}} = K$ a.s. (the left side is the closure of the set $\{X_n\}$)

Problem 265. Let X_n be independent and $\mathbf{P}(X_n = n^a) = \frac{1}{2} = \mathbf{P}(X_n = -n^a)$ where $a > 0$ is fixed. For what values of a does the series $\sum X_n$ converge a.s.? For which values of a does the series converge absolutely, a.s.?

Problem 266. Let $U_1, U_2, \dots$ be i.i.d. Unif$[0,1]$ random variables. Let $\xi_{i,j} = \mathbf{1}_{U_i < U_j}$. Show that $\sigma(\{\xi_{i,j}\}) = \sigma(\{U_k\})$. [*Note:* That is, we can recover the actual values of U_ks by just knowing the relative ordering among them!]

Problem 267. (**Random series**) Let X_n be i.i.d $N(0,1)$ for $n \geq 1$.

(1) Show that the random series $\sum X_n \frac{\sin(n\pi t)}{n}$ converges a.s., for any $t \in \mathbb{R}$.

(2) Show that the random series $\sum X_n \frac{t^n}{\sqrt{n!}}$ converges for all $t \in \mathbb{R}$, a.s.

[**Note:** The location of the phrase "a.s" is all important here. Let A_t and B_t denote the event that the series converges for the fixed t in the first or second parts of the question, respectively. Then, the first part is asking you to show that $\mathbf{P}(A_t) = 1$ for each $t \in \mathbb{R}$, while the second part is asking you to show that $\mathbf{P}(\cap_{t\in\mathbb{R}} B_t) = 1$. It is also true (and very important!) that $\mathbf{P}(\cap_{t\in\mathbb{R}} A_t) = 1$ but showing that is not easy.]

Problem 268. Suppose X_n are i.i.d random variables with finite mean. Which of the following assumptions guarantee that $\sum X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.

(2) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

Problem 269. (**Large deviation for Bernoullis**). Let X_n be i.i.d Ber$(1/2)$. Fix $p > \frac{1}{2}$.

(1) Show that $\mathbf{P}(S_n > np) \leq e^{-np\lambda} \left(\frac{e^\lambda + 1}{2}\right)^n$ for any $\lambda > 0$.

(2) Optimize over λ to get $\mathbf{P}(S_n > np) \leq e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the *entropy* of the Ber(p) measure introduced in the first class test).

(3) Recall that $S_n \sim$ Binom$(n, 1/2)$, to write $\mathbf{P}(S_n = \lceil np \rceil)$ and use Stirling's approximation to show that

$$\mathbf{P}(S_n \geq np) \geq \frac{1}{\sqrt{2\pi np(1-p)}} e^{-nI(p)}.$$

(4) Deduce that $\mathbf{P}(S_n \geq np) \approx e^{-nI(p)}$ for $p > \frac{1}{2}$ and $\mathbf{P}(S_n < np) \approx e^{-nI(p)}$ for $p < \frac{1}{2}$ where the notation $a_n \approx b_n$ means $\frac{\log a_n}{\log b_n} \to 1$ as $n \to \infty$ (i.e., asymptotic equality on the logarithmic scale).

Problem 270. Carry out the same program for i.i.d exponential(1) random variables and deduce that $\mathbf{P}(S_n > nt) \approx e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}(S_n < nt) \approx e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

Problem 271. Let $V = \frac{1}{\sqrt{n}}(Z_1, \ldots, Z_n)^t$ where $Z_1, \ldots, Z_n$ are i.i.d. $N(0,1)$ random variables. Show that

$$\mathbf{P}\{|\|V\| - 1| \geq t\} \leq 2e^{-c\frac{t^2}{n}}$$

for some $c > 0$ and all $t > 0$.

Problem 272. Let $Y_1, \ldots, Y_n$ be independent random variables. A random variable τ taking values in $\{1, 2, \ldots, n\}$ is called a *stopping time* if the event $\{\tau \leq k\} \in \sigma(Y_1, \ldots, Y_k)$ for all k (equivalently $\{\tau = k\} \in \sigma(Y_1, \ldots, Y_k)$ for all k).

(1) Which of the following are stopping times? $\tau_1 := \min\{k \leq n : S_k \in A\}$ (for some fixed $A \subseteq \mathbb{R}$). $\tau_2 := \max\{k \leq n : S_k \in A\}$. $\tau_3 := \min\{k \leq n : S_k = \max_{j \leq n} S_j\}$. In the first two cases set $\tau = n$ if the desired event does not occur.

(2) Assuming each X_k has zero mean, show that $\mathbf{E}[S_\tau] = 0$ for any stopping time τ. Assuming that each X_k has zero mean and finite variance, show that $\mathbf{E}[S_1^2] \leq \mathbf{E}[S_\tau^2] \leq \mathbf{E}[S_n^2]$ for any stopping time τ.

(3) Give examples of random τ that are not stopping times and for which the results in the second part of the question fail.

Problem 273. For each of the following statements, state whether they are true or false, and justify or give counterexample accordingly.

(1) If $\sum_n X_n$ converges a.s. and $\mathbf{P}(Y_n = X_n) = 1 - \frac{1}{n^2}$. Then $\sum_n Y_n$ converges a.s.

(2) If $\{X_n\}$ is an L^2 bounded sequence of random variables, and $\mathbf{E}[X_n] = 1$ for all n, then X_n cannot converge to zero in probability.

(3) If $X_n \xrightarrow{d} X$, then $X_n^2 \xrightarrow{d} X^2$.

(4) Suppose X_n are independent with $\mathbf{E}[X_n] = 0$ and $\sum \mathrm{Var}(X_n) = \infty$. Then, almost surely $\sum X_n$ does not converge.

(5) Suppose X_n, Y_n are random variables such that $|X_n| \leq |Y_n|$ for all n. If $\sum Y_n$ converges almost surely, then $\sum X_n$ converges almost surely.

Problem 274. Let X_k be independent random variables with zero mean and unit variance. Assume that $\mathbf{E}[|X_k|^{2+\delta}] \le M$ for some $\delta < 0$ and $M < \infty$. Show that S_n is asymptotically normal.

Problem 275. Let X_n be independent random variables with $X_n = \pm\sqrt{n}$ with probability 1/2. Show that S_n satisfies the central limit theorem but not the law of large numbers.

Problem 276. A manufacturer of nails packages them in boxes that say "1000 nails". On average 5 out of 1000 nails are defective and customers complain if there are fewer than 1000 non-defective nails in the box. To reduce the complaints to below 1% of the customers, the manufacturer puts m extra nails in each box. What is the minimum vale of m?

(1) Do it using CLT.

(2) Do it using just Chebyshev inequality.

Problem 277. Fix $\alpha > 0$.

(1) If X, Y are i.i.d. random variables such that $\frac{X+Y}{2^{\frac{1}{\alpha}}} \overset{d}{=} X$, then show that X must have characteristic function $\varphi_X(\lambda) = e^{-c|\lambda|^\alpha}$ for some constant c.

(2) Show that for $\alpha = 2$ we get $N(0, \sigma^2)$ and for $\alpha = 1$ we get symmetric Cauchy.

[*Note:* Only for $0 < \alpha \le 2$ is $e^{-c|\lambda|^\alpha}$ a characteristic function. Hence a distribution with the desired property exists only for this range of α].

Problem 278. Suppose X, Y are i.i.d. and $\frac{X+Y}{2^{1/\alpha}} \overset{d}{=} X$.

(1) If $0 < \text{Var}(X) < \infty$, show that $\alpha = 2$ and $X \sim N(0, \sigma^2)$ for some $\sigma^2 \ge 0$.

(2) If X has characteristic function $e^{-c|t|^\alpha}$ with $\alpha > 2$, deduce that $\text{Var}(X) < \infty$ and conclude that $X = 0$ (i.e., Stable-α distributions do not exist for $\alpha > 2$).

Problem 279. Let X_k be independent $\text{Ber}(p_k)$ random variables. If $\text{Var}(S_n)$ stays bounded, show that S_n cannot be asymptotically normal.

Problem 280. Let X_n be independent random variables with zero mean and unit variance. If $\{X_n^2\}$ is uniformly integrable, show that $\frac{S_n}{\sqrt{n}} \overset{d}{\to} N(0, 1)$.

Problem 281. Let $U_1, U_2, \ldots$ be i.i.d. uniform$[0, 1]$ random variables. Fix $0 < q < 1$ and let $M_n^{(q)}$ be the qth quantile, i.e., the $\lfloor nq \rfloor$th largest of the X_is (e.g., if $q = 1/2$, this is essentially the median). Show that $\sqrt{n}(M_n^{(q)} - q) \overset{d}{\to} N(0, q(1 - q))$.

Problem 282. A simple model for grinding particles down: Start with a particle of size 1. After one cycle of grinding, it breaks into two particles of sizes X and $1-X$, where $X \sim \mu$, a non-degenerate probability measure on $[0,1]$. Each particle of size s similarly breaks into two particles of sizes Ys and $(1-Y)s$, where $Y \sim \mu$. The random variables indicating the breaking proportion are assumed independent.

If the particle sizes are $X_{n,j}$, $j \le 2^n$, after n cycles of grinding, show that the proportion of j for which $\sqrt{n}\log X_{n,j} \le t$ converges to $\mathbf{P}\{Z \le t\}$ where $Z \sim N(0,1)$.

[*Note:* Perhaps easier, show the same for the *expected* proportion of j for which $\sqrt{n}\log X_{n,j} \le t$. This problem is a simplification of a model first proposed by Kolmogorov, where he allows each particle to subdivide into an arbitrary number of particles.]

Problem 283. Out of the $n!$ permutations of the set $[n] = \{1,2,\ldots,n\}$, pick one at random and call it Π. Let C_n be the number of cycles in the cycle decomposition of Π.

(1) Define A_k be the event that k is the lowest element in its cycle. Show that $A_1,\ldots,A_n$ are *independent* and that $\mathbf{P}(A_k) = (n-k+1)/n$.

(2) Show that $\frac{C_n}{\log n} \xrightarrow{P} 1$.

(3) Show that $\frac{C_n - \log n}{\sqrt{\log n}} \xrightarrow{d} N(0,1)$.

Problem 284. Out of the $n!$ permutations of the set $[n] = \{1,2,\ldots,n\}$, pick one at random and call it Π. Let $\mathcal{I}_n$ denote the number of inversions of Π, i.e., the number of pairs $i<j$ such that $\Pi(i) > \Pi(j)$. Show that

$$\frac{\mathcal{I}_n - \frac{n(n-1)}{4}}{\sqrt{n^3/36}} \xrightarrow{d} N(0,1).$$

Problem 285. Let X_n be independent, and let $X_n \sim (\frac{1}{2} - 2\varepsilon_n)\delta_{\pm 1} + \varepsilon_n \delta_{\pm M_n}$ where $\varepsilon_n \downarrow 0$ and $M_n \uparrow \infty$.

(1) Find a condition on M_n, ε_n that allows to apply Lindeberg-Feller theorem directly to prove that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$.

(2) If $\sum_n \varepsilon_n < \infty$, show that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$ even if M_n are chosen to violate the condition in the first part.

Problem 286. Produce an example of independent random variables X_n so that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$, but $\text{Var}(S_n/\sqrt{n}) \to 2$. Can you make $\text{Var}(S_n/\sqrt{n}) \to \infty$?

Problem 287. Suppose X_n are independent random variables taking values ± 1 with probability n^{-b} each and taking the value 0 with probability $1-2n^{-b}$. Here $b>0$.

(1) Find the range of b for which $\sum_{n=1}^{\infty} X_n$ converges almost surely.

(2) Find the range of b for which $\frac{S_n}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{d} N(0,1)$.

Problem 288. Let X_n be i.i.d. random variables with zero mean and unit variance. Let ε_n be independent of X_js and among themselves and $\mathbf{P}\{\varepsilon_n = \pm 1\} = \frac{1}{2n^2}$ and $\mathbf{P}\{\varepsilon_n = 0\} = 1 - \frac{1}{n^2}$. Set $Y_n = (1-\varepsilon_n)X_n + \varepsilon_n n$.

(1) Show that $\frac{S_n^Y}{\sqrt{n}} \xrightarrow{d} N(0,1)$ by comparing with S_n^X.

(2) Show that $\mathrm{Var}(\frac{S_n^Y}{\sqrt{n}}) \to 2$.

Problem 289. Let U_k, V_k be i.i.d Uniform([0,1]) random variable.

(1) Show that $\sum_k U_k^{\frac{1}{k}} - V_k^{\frac{1}{k}}$ converges a.s.

(2) Let $S_n = U_1 + U_2^2 + \ldots + U_n^n$. Show that S_n satisfies a CLT. In other words, find a_n, b_n such that $\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$.

Problem 290 (*Weak law using characteristic functions)***.** Let X_k be i.i.d. random variables having characteristic function φ.

(1) If $\varphi'(0) = i\mu$, show that the characteristic function of S_n/n converges to the characteristic function of δ_μ. Conclude that weak law holds for S_n/n.

(2) If $\frac{1}{n}S_n \xrightarrow{P} \mu$ for some μ, then show that φ is differentiable at 0 and $\varphi'(0) = i\mu$.

Problem 291. Find the characteristic functions of the distributions with the given densities.
(1) $e^{-|x|}$ for $x \in \mathbb{R}$, (2) $\frac{1}{2}\left(1 - \frac{|x|}{2}\right)_+$.

Problem 292. Find the distributions whose characteristic functions are (1) $t \mapsto \cos(t)$, (2) $t \mapsto \frac{1}{1+it}$.

Problem 293. Show $\frac{1}{2}\mathrm{sech}(\frac{\pi x}{2})dx$ is a probability measure whose characteristic function is $\mathrm{sech}(t)$.

Problem 294. Show that the characteristic function of the arcsine measure having density $\frac{1}{\pi\sqrt{1-x^2}}$ on $[-1,1]$ is equal to the Bessel function $J_0(t) = \int_{-\pi}^{\pi} e^{-it\sin\theta} \frac{d\theta}{2\pi}$.

Problem 295. Use characteristic functions to show that the sum of independent Poisson random variables is Poisson and other such statements for Binomial, Normal, Exponential/Gamma, Cauchy.

Problem 296. If $x_n \in \mathbb{R}$ and $e^{itx_n} \to 1$ for all $t \in \mathbb{R}$, then show that $x_n \to 0$.

Problem 297. Suppose $\mu \in \mathcal{P}(\mathbb{R})$. Suppose that $|\hat{\mu}(a)| = 1$ for some $a \neq 0$. Show that there is a $\delta > 0$ such that $\mu(\delta\mathbb{Z}) = 1$.

Problem 298. If φ is a characteristic function, show that the following are also characteristic functions as a function of t. (1) $|\varphi(t)|^2$, (2) $e^{\varphi(t)-1}$, (3) $\frac{1}{t}\int_0^t \varphi(s)ds$, (4) $\varphi(t)\varphi(2t)$, (5) $\frac{1}{1-\frac{1}{2}\varphi(t)}$.

Problem 299. If φ is a smooth characteristic function, show that $\varphi''(0) \leq 0$ (in particular it is real-valued). Can equality hold?

Problem 300. If μ is a probability measure, show that

$$\mu\{x_0\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \hat{\mu}(t)e^{-itx_0}dx.$$

Problem 301. Suppose μ_n, μ are probability measures on $\mathbb{R}$ with characteristic functions φ_n, φ. If $\varphi_n(t) \to \varphi(t)$ for all $t \in \mathbb{Q}$, is it true that $\mu_n \to \mu$ weakly?

Problem 302. If ψ is a real-valued characteristic function, show that

$$1 - \psi(2t) \leq 4(1 - \psi(t)).$$

Deduce that if φ is any characteristic function, then

$$1 - |\varphi(2t)| \leq 8(1 - |\varphi(t)|).$$

Problem 303. A random variable X has characteristic function

$$\exp\left\{\sum_{j=1}^{n} \theta_j(e^{itx_j} - 1 - itx_j)\right\}$$

for some $x_i \in \mathbb{R}$ and $\theta_i > 0$. Describe/construct X in terms of familiar random variables.

Problem 304. Let ψ be the characteristic function of X.

(1) Fix $t_1, \ldots, t_n$ and $c_1, \ldots, c_n$ and find $\mathbf{E}[|Y|^2]$ where $Y = c_1e^{it_1X} + \ldots + c_ne^{it_nX}$.

(2) Use the first part to argue that ψ is positive definite in the sense that the matrix $(\psi(t_k - t_j))_{j,k\le n}$ is positive semi-definite, for any $t_1, \dots t_n$.

Problem 305. Let $\mu \in \mathcal{P}(\mathbb{R})$ and suppose $|\hat{\mu}(t_0)| = 1$ for some $t_0 \neq 0$. Then, μ is supported on a lattice, that is, $\mu(a\mathbb{Z} + b) = 1$ for some $a, b \in \mathbb{R}$.

Problem 306. Show that $\psi(t) = e^{-|t|^\alpha}$ is not a characteristic function if $\alpha > 2$. [*Hint:* Use the previous exercise with $n = 3$ and suitably chosen t_js]

Problem 307. Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a characteristic function that satisfies (a) $\varphi(0) = 1$, (b) $\varphi(t) = 0$ for $t \ge 2$, (c) φ is even, (d) φ is linear on $[0, 1]$ and on $[1, 2]$.If $t_0 \in [0, 1]$, show that the possible values of $\varphi(t_0) \le 1 - t_0$.

Problem 308. Give another proof that $\psi(t) = e^{-|t|^\alpha}$ is not a characteristic function if $\alpha > 2$.

(1) Assuming that ψ is a characteristic function of X, show that $\mathrm{Var}(X) < \infty$.

(2) Show that $\frac{X+Y}{2^{1/\alpha}} \stackrel{d}{=} X$, where Y is an independent copy of X.

Get a contradiction using these two statements.

Problem 309. Let $X \sim \mu$ be a random variable with characteristic function φ. Show that the following are equivalent.

(1) $X \stackrel{d}{=} Y_1 + Y_2$ for some i.i.d. random variables Y_1, Y_2.

(2) $\varphi = \psi^2$ for a characteristic function ψ.

Problem 310. Show that there are independent X, Y, Z such that $X + Y \stackrel{d}{=} X + Z$ but $Y \stackrel{d}{\neq} Z$ (so you cannot "cancel" X on both sides).

Problem 311. Let μ be a probability measure with non-negative characteristic function $\hat{\mu} \ge 0$.

(1) If μ is supported on integers, show that $\mu\{0\} \ge \mu\{k\}$ for all $k \in \mathbb{Z}$.

(2) If $\hat{\mu}$ is integrable, show that the density of μ exists and attains its maximum at 0.

Problem 312. Let $U \sim \mathrm{Unif}[-1, 1]$. Use your knowledge of its characteristic function and of its binary digits to show the identity

$$\frac{\sin(t)}{t} = \prod_{n=1}^{\infty} \cos(t/2^n).$$

Problem 313. Let ξ_n be i.i.d. $\text{Ber}_{\pm}(1/2)$ and let $X_\lambda = \sum_{n=1}^{\infty} \xi_n \lambda^n$, where $0 < \lambda < 1$.

(1) Show that X_λ has characteristic function $\psi(t) = \prod_{n=1}^{\infty} \cos(t\lambda^n)$.

(2) If $d(\lambda^{-n}, \mathbb{Z}) \to 0$, show that $\psi(2\pi/\lambda^n) \not\to 0$ and hence deduce that X_λ has no density.

(3) Check that $\lambda = \frac{2}{1+\sqrt{5}}$ (satisfies $\lambda^{-2} - \lambda^{-1} - 1 = 0$) is one such number.

Problem 314. Here are some integral identities. Prove them using characteristic functions!

(1) $\int_{\mathbb{R}} e^{-\frac{1}{2}t^2} \frac{\sin t}{t} dt = \frac{\sqrt{\pi}}{\sqrt{2}} \int_{-1}^{1} e^{-\frac{1}{2}x^2} dx$.

(2) $\int_{\mathbb{R}} \frac{(\sin t)^2}{t^2} dt = \pi$.

(3) $\int_{\mathbb{R}} \frac{1}{(1+t^2)^2} dt = \frac{\pi}{4}$.

Problem 315. Let μ_n, μ be probability measures. If $\hat{\mu}_n$ converges uniformly to $\hat{\mu}$, then $F_{\mu_n} \to F_\mu$ uniformly on $\mathbb{R}$. The following steps are suggested.

(1) If F_μ is continuous, see Problem 55.

(2) If μ is discrete (start with $\mu = \delta_0$), use Problem 300.

(3) For general μ, separate into the discrete and continuous parts.

Problem 316. Let μ be a probability measure with C^1 density f. Show that $\int_{\mathbb{R}} |\hat{\mu}|^2 < \infty$ and that $\int_{\mathbb{R}} \hat{\mu}(t)^2 dt = \pi \int_{\mathbb{R}} f(x)f(-x)dx$ and $\int_{\mathbb{R}} |\hat{\mu}(t)|^2 dt = \pi \int_{\mathbb{R}} f(x)^2 dx$.

Problem 317. Let μ be a probability measure on $\mathbb{R}$. If $\hat{\mu}(t) \geq 0$ for all t and $\int |\hat{\mu}(t)| dt < \infty$, then show that μ has a continuous density f and that $\sup_x f(x) = f(0)$.

Problem 318. Let X_n be i.i.d. random variables with a non-degenerate distribution. If $S_n = X_1 + \ldots + X_n$, show that $\mathbf{P}\{|S_n| \leq M\} \to 0$ for any $M < \infty$.

Problem 319. Let $Z \sim N(0, 1)$.

(1) Show that $\mathbf{E}[e^{nZ} \sin(\pi m Z)] = 0$ for all $n, p \in \mathbb{Z}$.

(2) Conclude that if $X = e^Z$ and $Y = e^Z \sin(\pi Z)$, then $\mathbf{E}[X^n] = \mathbf{E}[Y^n]$ for all $n \in \mathbb{N}$.

(3) Show that X and Y do not have the same distribution.

[*Note*: It may be helpful to write sin in terms of complex exponential. The point is that there are two random variables with different distributions that have identical moments]

Remark (for the next three problems): The characteristic function of a $\mathbb{R}^d$-valued random vector X is the function $u \mapsto \mathbf{E}[e^{i\langle u,X\rangle}]$ from $\mathbb{R}^d \to \mathbb{C}$. Assume the following facts: If X and Y have the same characteristic functions, then $X \stackrel{d}{=} Y$. If $\mathbf{E}[e^{i\langle u,X_n\rangle}] \to \mathbf{E}[e^{i\langle u,X\rangle}]$ for all $u \in \mathbb{R}^d$, then $X_n \stackrel{d}{\to} X$.

Problem 320. Show that the measures of half-spaces (i.e., $\mathbf{P}\{\langle X, v\rangle \le r\}$, where $v \in \mathbb{R}^d$, $r \in \mathbb{R}$) determine the distribution of X. Similarly, show that if $\langle X_n, v\rangle \stackrel{d}{\to} \langle X, v\rangle$ for each $v \in \mathbb{R}^d$, then $X_n \stackrel{d}{\to} X$.

Problem 321. If X_n are independent random vectors in $\mathbb{R}^d$ with $\mathbf{E}[X_n] = 0$ and $\mathbf{E}[X_n X_n^t] = \Sigma$, then show that $\frac{S_n}{\sqrt{n}} \stackrel{d}{\to} N_d(0,\Sigma)$, which is the defined as the distribution with the characteristic function $t \mapsto e^{-\frac{1}{2}u^t\Sigma u}$.

Problem 322. If Σ is invertible, show that $N_d(0,\Sigma)$ has density $\frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}}e^{-\frac{1}{2}x^t\Sigma^{-1}x}$.

Problem 323. Let $\mathbf{Z}^{(n)} = (Z_1^{(n)}, \ldots, Z_n^{(n)})$ be a point sampled uniformly from the sphere S^{n-1} (this means that $\mathbf{P}(\mathbf{Z}^{(n)} \in A) = \text{area}(A)/\text{area}(S^{n-1})$ for any Borel set $A \subseteq S^{n-1}$).

(1) Find the density of $Z_1^{(n)}$.

(2) Using (1) or otherwise, show that $\sqrt{n}Z_1^{(n)} \stackrel{d}{\to} N(0,1)$ as $n \to \infty$.

[**Hint:** One way to generate $\mathbf{Z}^{(n)}$ is to sample $X_k \sim N(0,1)$ i.i.d., and to set $\mathbf{Z}^{(n)} = \frac{1}{\|X\|}(X_1, \ldots, X_n)$ where $\|X\| = \sqrt{X_1^2 + X_2^2 + \ldots + X_n^2}$. You may assume this fact without having to justify it].