FINAL EXAM: PROBABILITY THEORY

26TH APRIL, 10AM-1:00PM

MAXIMUM MARKS: 50, DURATION: 180 MINUTES

Note: *Give all relevant justifications but write succinctly and legibly. Ask questions only if there is an ambiguity in the question and not to verify your answers.*

1. [4 marks each] For each of the following statements, state whether they are true or false, and justify or give counterexample accordingly.

(1) Let $\mu_n \in \mathcal{P}(\mathbb{R})$. If $F_{\mu_n}(t) \to F(t)$ for all t for some function $F : \mathbb{R} \to (0,1)$, then F is necessarily the CDF of a Borel probability measure on $\mathbb{R}$.

(2) If μ, ν, θ are Borel probability measures on $\mathbb{R}$ and $\mu \perp \nu$ and $\nu \perp \theta$ and $\mu \perp \theta$, then there exist pairwise disjoint sets $A, B, C \in \mathcal{B}(\mathbb{R})$ such that $\mu(A) = 1$, $\nu(B) = 1$ and $\theta(C) = 1$.

(3) If $\mu, \nu \in \mathcal{P}(\mathbb{R})$, then there exists a Borel measurable $T : \mathbb{R} \to \mathbb{R}$ such that $\mu \circ T^{-1} = \nu$.

(4) Let X_n be independent random variables with $\mathbf{E}[X_n] = 0$ and $\text{Var}(X_n) \leq 1$ for all n. Let $T_n = (X_1 + \ldots + X_n)/\sqrt{n}$. Then, $\{T_n\}$ is tight.

(5) If X_n are independent random variables and $X_n \xrightarrow{P} X$, then X is a constant, $a.s.$

(6) If $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{L^q} X$ for any $q \in (0, p)$.

2. [5 marks] Suppose $X_n \geq 0$, $\mathbf{E}[X_n] = 1$ for all n, and $X_n \xrightarrow{a.s.} X$. Show that the possible values of $\mathbf{E}[X]$ are all numbers in $[0,1]$.

3. [5 marks] Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any event $A \in \mathcal{F}$ with $\mathbf{P}(A) < \delta$, we have $\mathbf{E}[X\mathbf{1}_A] < \epsilon$.

4. [5 marks] On the probability space $([0,1], \mathcal{B}, \lambda)$, define the random variables $X(t) = \sin(2\pi t)$ and $Y(t) = \cos(2\pi t)$.

(1) Describe the sigma-algebra generated by X and the sigma-algebra generated by X and Y.

(2) Are X and Y independent?

5. [**5 marks**] Let X_n be i.i.d. unif$[-1,1]$ and let $a_n > 0$ be given numbers such that $\sum_{n=1}^{\infty} a_n^2 = \infty$. Let $T_n = \sum_{k=1}^{n} a_k X_k$. Show that $\{T_n\}$ is asymptotically normal, i.e., $\frac{1}{\sigma_n}(T_n - \mu_n) \to N(0,1)$ for some (non-random) numbers $\mu_n \in \mathbb{R}$ and $\sigma_n > 0$.

6. [**5 marks**] Let X_k be i.i.d. Exp(1) random variables and let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $\frac{1}{\log n} M_n \xrightarrow{P} 1$.

7. [**5 marks**] Suppose X_n are i.i.d., symmetric random variables such that $|X_1| \leq 1$ $a.s.$ Show that $n^{-\gamma} S_n \xrightarrow{a.s.} 0$ for any $\gamma > \frac{1}{2}$.