Homework 3: Due 17th Feb

Submit the first four problems only

1. Let X be a random variable with distribution μ and X_n are random variables defined as follows. If μ_n is the distribution of X_n, in each case, show that $\mu_n \xrightarrow{d} \mu$ as $n \to \infty$.

(1) (Truncation). $X_n = (X \wedge n) \vee (-n)$.

(2) (Discretization). $X_n = \frac{1}{n}\lfloor nX \rfloor$.

2. (1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.

(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda > 0\}$ tight?

3. Let F be a CDF on $\mathbb{R}$.

(1) Show that F can have at most countably many discontinuity points. [**Hint:** How many jumps of size more than $1/10$ can it have?]

(2) Give example of a CDF that has a dense set of discontinuity points.

4. Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that their distribution functions are continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$.

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Do not submit the following problems but highly recommended to try them

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5. Consider the family of Normal distributions, $\{N(\mu,\sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$. Show that the map $(\mu,\sigma^2) \to N(\mu,\sigma^2)$ from $\mathbb{R} \times \mathbb{R}_+$ to $\mathcal{P}(\mathbb{R})$ is continuous. (Complicated way of saying that if $(\mu_n,\sigma_n^2) \to (\mu,\sigma^2)$, then $N(\mu_n,\sigma_n^2) \xrightarrow{d} N(\mu,\sigma^2)$).

Do the same for other natural families if distributions, (1) Exp(λ), (2) Uniform$[a,b]$, (3) Bin(n,p) (fix n and show continuity in p), (4) Pois(λ).

6. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

7. Consider the space $X = [0,1]^{\mathbb{N}} := \{\mathbf{x} = (x(1), x(2), \ldots) : 0 \le x(i) \le 1 \text{ for each } i \in \mathbb{N}\}$. Define the metric $d(\mathbf{x},\mathbf{y}) = \sup_i \frac{|x(i)-y(i)|}{i}$.

(1) Show that $\mathbf{x}_n \to \mathbf{x}$ in (X,d) if and only if $x_n(i) \to x(i)$ for each i, as $n \to \infty$.

[**Note:** What matters is this pointwise convergence criterion, not the specific metric. The resulting topology is called *product topology*. The same convergence would hold if we had defined the metric as $d(\mathbf{x},\mathbf{y}) = \sum_i 2^{-i}|x(i)-y(i)|$ or $d(\mathbf{x},\mathbf{y}) = \sum_i i^{-2}|x(i)-y(i)|$ etc., But not the metric $\sup_i |x(i)-y(i)|$ as convergence in this metric is equivalent to uniform convergence over all $i \in \mathbb{N}$].

(2) Show that X is compact.

[**Note:** What is this problem doing here? The purpose is to reiterate a key technique we used in the proof of Helly's selection principle!]

8. Recall the Cantor set $C = \bigcap_n K_n$ where $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, etc. In general, $K_n = \bigcup_{1 \le j \le 2^n} [a_{n,j}, b_{n,j}]$ where $b_{n,j} - a_{n,j} = 3^{-n}$ for each j.

(1) Let μ_n be the uniform probability measure on K_n. Describe its CDF F_n.

(2) Show that F_n converges uniformly to a CDF F.

(3) Let μ be the probability measure with CDF equal to F. Show that $\mu(C) = 1$.

9. Let $\mu \in \mathcal{P}(\mathbb{R})$.

(1) For any $n \ge 1$, define a new probability measure by $\mu_n(A) = \mu(n.A)$ where $n.A = \{nx : x \in A\}$. Does μ_n converge as $n \to \infty$?

(2) Let μ_n be defined by its CDF

$$F_n(t) = \begin{cases} 0 & \text{if } t < -n, \\ F(t) \text{ if } -n \le t < n, & \\ 1 & \text{if } t \ge n. \end{cases}$$

Does μ_n converge as $n \to \infty$?

(3) In each of the cases, describe μ_n in terms of random variables. That is, if X has distribution μ, describe a transformation $T_n(X)$ that has the distribution μ_n.

10. Show that under the Lévy metric, $\mathcal{P}(\mathbb{R})$ is a complete and seperable metric space.