Homework 5: Due 20th Mar

Submit the first four problems only

1. (1) If X, Y are independent random variables, show that $\mathrm{Cov}(X, Y) = 0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1, X_2, \ldots$ such that $\mathrm{Cov}(X_i, X_j) = 0$ for any $i \neq j$ but such that X_i are not independent.

2. (1) Suppose $2 \leq k < n$. Give an example of n random variables $X_1, \ldots, X_n$ such that any subset of k of these random variables are independent but no subset of $k+1$ of them is independent.

(2) Suppose $(X_1, \ldots, X_n)$ has a multivariate Normal distribution. Show that if X_i are pairwise independent, then they are independent.

3. Suppose $(X_1, \ldots, X_n)$ has density f (w.r.t Lebesgue measure on $\mathbb{R}^n$).

(1) If $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions g_k, $k \leq n$. Then show that $X_1, \ldots, X_n$ are independent. (Don't assume that g_k is a density!)

(2) If $X_1, \ldots, X_n$ are independent, then $f(x_1, \ldots, x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1, \ldots, g_n$.

4. If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

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Do not submit the following problems but recommended to try them or at least read them!

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5. Let X_i, $i \geq 1$ be random variables on a common probability space. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable function (with product sigma algebra on $\mathbb{R}^{\mathbb{N}}$ and Borel sigma algebra on $\mathbb{R}$) and let $Y = f(X_1, X_2, \ldots)$. Show that the distribution of Y depends only on the joint distribution of $(X_1, X_2, \ldots)$ and not on the original probability space. [**Hint:** We used this idea to say that if X_i are independent Bernoulli random variables, then $\sum_{i \geq 1} X_i 2^{-i}$ has uniform distribution on $[0, 1]$, irrespective of the underlying probability space.]

6. Let $\mathcal{G}$ be the countable-cocountable sigma algebra on $\mathbb{R}$. Define the probability measure μ on $\mathcal{G}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable. Show that μ is *not* the push-forward of Lebesgue measure on $[0,1]$, i.e., there does not exist a measurable function $T : [0,1] \mapsto \Omega$ (w.r.t. the σ-algebras $\mathcal{B}$ and $\mathcal{G}$) such that $\mu = \lambda \circ T^{-1}$.

7. Show that it is not possible to define uncountably many independent Ber$(1/2)$ random variables on the probability space $([0,1], \mathcal{B}, \lambda)$.

8 (Existence of Markov chains). Let S be a countable set (with the power set sigma algebra). Two ingredients are given: (1) *A transition matrix*, that is, a function $p : S \times S \to [0,1]$ be a function such that $p(x, \cdot)$ is a probability mass function on S for each $x \in S$. (2) *An initial distribution*, that is a probability mass function μ_0 on S.

For $n \geq 0$ define the probability mass function ν_n on S^{n+1} (with the product sigma algebra) by

$$\nu_n(x_0, \ldots, x_n) = \mu_0(x_0) \prod_{j=0}^{n-1} p(x_j, x_{j+1}).$$

Show that ν_n is a valid probability mass function and that they form a consistent family. Conclude that a Markov chain with initial distribution μ_0 and transition matrix p exists.