HOMEWORK 8: DUE 26TH APR

SUBMIT THE FIRST ZERO PROBLEMS ONLY

1. Let X_k be independent random variables with zero mean and unit variance. Assume that $\mathbf{E}[|X_k|^{2+\delta}] \leq M$ for some $\delta < 0$ and $M < \infty$. Show that S_n is asymptotically normal.

2. Let X_k be i.i.d. random variables with zero mean and unit variance. Let $0 < a_1 < a_2 < \ldots$ be given numbers. Find sufficient conditions on $(a_i)_i$ such that S_n is asymptotically normal.

3. Let X_k be independent Ber(p_k) random variables. If Var(S_n) stays bounded, show that S_n cannot be asymptotically normal.

4. Fix $\alpha > 0$.

(1) If X, Y are i.i.d. random variables such that $\frac{X+Y}{2^{\frac{1}{\alpha}}} \stackrel{d}{=} X$, then show that X must have characteristic function $\varphi_X(\lambda) = e^{-c|\lambda|^\alpha}$ for some constant c.

(2) Show that for $\alpha = 2$ we get $N(0, \sigma^2)$ and for $\alpha = 1$ we get symmetric Cauchy.

[*Note:* Only for $0 < \alpha \leq 2$ is $e^{-c|\lambda|^\alpha}$ a characteristic function. Hence a distribution with the desired property exists only for this range of α].

5 (*Weak law using characteristic functions*)**.** Let X_k be i.i.d. random variables having characteristic function φ.

(1) If $\varphi'(0) = i\mu$, show that the characteristic function of S_n/n converges to the characteristic function of δ_μ. Conclude that weak law holds for S_n/n.

(2) If $\frac{1}{n}S_n \stackrel{P}{\to} \mu$ for some μ, then show that φ is differentiable at 0 and $\varphi'(0) = i\mu$.

6. Find the characteristic functions of the distributions with the given densities. (1) $e^{-|x|}$ for $x \in \mathbb{R}$, (2) $\frac{1}{2}\left(1 - \frac{|x|}{2}\right)_+$.

7 (*Multidimensional central limit theorem*)**.** Let X_n be i.i.d. $\mathbb{R}^d$-valued random vectors with zero mean and covariance matrix Σ. Let $S_n = X_1 + \ldots + X_n$. Show that $\frac{1}{\sqrt{n}}S_n \stackrel{d}{\to} N_d(0, \Sigma)$ using the replacement principle. Assume (for convenience) that third moments are finite (i.e., $\mathbf{E}[\|X_1\|^3] < \infty$).