Problem set 1

Due date: **16th Aug**

Submit the starred exercises only

Exercise 1 (*). For each of the "random experiments" described below, describe the sample space and the probabilities. Also compute the probability of the event A specified. If no event is specified, just give the probability space.

(1) A fair die is thrown until a 6 or a 1 shows up. Let A be the event that the number of throws is at least n.
(2) A coin is tossed and a die is thrown. A is the event that either the coin turns up head or the die shows up an even number.
(3) Place k unlabeled balls in n labelled urns. Let A be the event that the first urn is empty. [**Note:** See the next part before you think you have solved this.]
(4) Place k unlabeled balls in n labelled urns so that all *distinguishable configurations* are equally likely. Let A be the event that the first urn is empty.

Exercise 2 (*). Let $A_1,\ldots,A_n$ be events in a probability space (Ω, p) and let $m \le n$. Let B_m be the event that at least m of the events $A_1,\ldots,A_n$ occur. That is

$$B_m = \bigcup_{1\le i_1<i_2<\ldots<i_m\le n} (A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_m}).$$

In class we showed by the inclusion exclusion formula that $\mathbf{P}(B_1) = S_1 - S_2 + \ldots + (-1)^{n-1}S_n$ where

$$S_k = \sum_{1\le i_1<i_2<\ldots<i_k\le n} \mathbf{P}(A_{i_1}\cap A_{i_2}\ldots\cap A_{i_k}).$$

Show that

$$\mathbf{P}(B_m) = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \ldots + (-1)^{n-m}\binom{n-1}{m-1}S_n.$$

[**Note:** Also see Problem 4 and the remark at the end of it.]

Exercise 3. For each of the "random experiments" described below, describe the sample space and the probabilities. Also compute the probability of the event A specified. If no event is specified, just give the probability space.

(1) A die is thrown until the first time a head is immediately followed by a tail (eg., if the tosses are $TTHHT$ then we needed 5 tosses). Find the probability that at least n throws are needed.
(2) Place k labeled balls in n unlabeled urns. Let A be the event that the first ball and the second ball are in distinct urns. Do it for both cases - (a) Each *distinguishable* distribution is equally likely and (b) All distributions are equally likely (even if not distinguishable).
(3) Place k unlabeled balls in n unlabeled urns.

(4) 13 cards are dealt from a shuffled deck off 52 cards. Let A be the event that the cards dealt contains a series[1]. Let B be the event that the cards dealt contains a set[2].
(5) A drunkard returns home with a bunch of n keys in his pocket. He randomly tries them one after another till the lock opens. Let A be the event that the fifth key opens the lock. Assume that he does not try the same key twice.

Exercise 4. Let $A_1,\dots,A_n$ be events in a probability space (Ω, p) and let $m \le n$. Let C_m be the event that exactly m out of the n events $A_1,\dots,A_n$ occur. That is

$$C_m = \bigcup_{1\le i_1<i_2<\dots<i_m\le n} \left\{ \left(\bigcap_{j=1}^{m} A_{i_j} \right) \bigcap \left(\bigcap_{k\notin\{i_1,\dots,i_m\}} A_k^c \right) \right\}.$$

$$\mathbf{P}(C_m) = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \dots + (-1)^{n-m}\binom{n}{m} S_n.$$

[**Note:** If you solve one of Problem 2 or Problem 4, you can solve the other using the relationship $C_m = B_m \setminus B_{m+1}$ or $B_m = C_m \sqcup C_{m+1} \sqcup \dots \sqcup C_n$.]

[1] A series means three cards of the same suit in succession, eg., 9,10,J of spades. Here ace is interpreted as 1 and hence A,2,3 is a series but not QKA or KA2.

[2] A set means three cards of distinct suits but having the same value, eg., the queen of spades, diamonds and hearts or the 7 of spades, hearts and clubs.