Problem set 2

Due date: **2nd Sep**

Submit the starred exercises only

Exercise 5 (*)**.** A total of $n \geq 365$ people come to a party one after another. As they come in, their birthdays (date and month) are noted. We are interested in the first time that a birthday repeats. For simplicity assume there are 365 possible birthdays (i.e., exclude February 29th as a possible birthday).

(1) Write the probability space and the corresponding random variable.
(2) Find the distribution of the random variable under consideration.
(3) Find the *median* of this random variable. In other words, find the value of k such that $\mathbf{P}(X \geq k) \geq 0.5 \geq \mathbf{P}(X > k)$.

Exercise 6 (*)**.** m decks of cards (having n cards each) are shuffled and kept beside each other. We say that there is a match at level k if the k^{th} cards in all the decks are the same. Consider the event that there are no matches at all.

(1) Write the probability space, the event of interest and find an expression for the probability of the event.
(2) Use the first two Bonferroni inequalities to give upper and lower bounds for the above probability. Find the values for $m = 2,3,4$ and $n = 25,50,100$.

Exercise 7 (*)**.** Toss a possibly biased coin till you get a Head for the second time (not necessarily on two consecutive tosses). Consider the random variable that counts the total number of tosses.

(1) Write the probability space and the corresponding random variable.
(2) Find the distribution of the random variable under consideration.

[*Extra, don't need to submit:* Can you generalize this to the number of tosses needed to get k heads, where $k \geq 1$ is a fixed integer? The resulting distribution is called *negative binomial distribution with parameters* k, p and is denoted $\text{NegBin}(k, p)$ (for $k = 1$ we get back the $\text{Geo}(p)$ distribution).]

Exercise 8. Decide for each case whether $\sum_{x \in S} f(x)$ is absolutely summable or not. You do not need to find the value of the sum.

(1) $S = \mathbb{Z}$, $f : S \to \mathbb{R}$ is defined by $f(n) = \frac{1}{n^{\alpha}}$ where α is a positive integer. The answer will depend on α.
(2) $S = \mathbb{N} \times \mathbb{N} = \{(m,n) : m,n \in \mathbb{N}\}$ and $f(m,n) = \frac{(-1)^{m+n}}{m^2+n^2}$.
(3) $S = \mathbb{Q} \cap (0,1)$ and $f(r) = \frac{1}{q}$ where $r = \frac{p}{q}$ with $p,q \in \mathbb{N}$ and have no common factor. What if $f(r) = \frac{1}{q^3}$?

Exercise 9. Let $f(n)$ be a sequence of real numbers Let $f_+(n) = f(n) \vee 0$ and $f_-(n) = (-f(n)) \vee 0$. Assume that $f(n) \to 0$ as $n \to \infty$. Let $A_+ = \sum_n f_+(n)$ and let $A_- = \sum_n f_-(n)$. In this exercise we guide you to a proof of the equivalence of the two facts stated in class (absolute convergence and the invariance of the sum under reordering).

(1) Assume that A_+ and A_- are finite. Let $\pi : \mathbb{N} \to \mathbb{N}$ be any bijection. Show that

$$\lim_{N\to\infty} \sum_{k=1}^{N} f_+(\pi(k)) = \lim_{N\to\infty} \sum_{k=1}^{N} f_+(k) \qquad \lim_{N\to\infty} \sum_{k=1}^{N} f_+(\pi(k)) = \lim_{N\to\infty} \sum_{k=1}^{N} f_+(k).$$

Conclude that $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(\pi(k)) = \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(k)$.

[**Hint:** To show the first conclusion, let $L' = \lim\limits_{N\to\infty} \sum_{k=1}^{N} f_+(\pi(k))$ and let $L = \lim\limits_{N\to\infty} \sum_{k=1}^{N} f_+(k)$. By definition of bijection, for any $N \geq 1$, there is some $N' \geq 1$ such that $\{1,2,\ldots,N\} \subseteq \{\pi(1),\ldots,\pi(N')\}$ and hence $\sum_{k=1}^{N'} f_+(\pi(k)) \geq \sum_{k=1}^{N} f_+(k)$].

(2) If only one of A_+ or A_- is finite, show that again $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(\pi(k)) = \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(k)$ (while the limit is $\pm\infty$ this time).

(3) Suppose $A_+ = +\infty$ and $A_- = +\infty$. Fix a real number α. Show that there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} f(\pi(k)) = \alpha$. [**Hint:** Collect enough and just enough positive numbers from the beginning of the sequence $(f(n))$ to exceed α. Then collect enough negative terms to make the sum less than α. Now take positive terms again to make the sum exceed α. Proceed in this way. Argue that the limit exists and is equal to α.]