Problem set 4

Due date: **6th Oct**

Submit the starred exercises only

Exercise 16. In each of the following situations, describe the joint distribution of X_1 and X_2.

(1) Toss a (not necessarily fair) coin 3 times. Let X_1 be the outcome of the first toss and let X_2 be the outcome of the second toss.
(2) In the same setting as above, let X_1 be the number of heads in the first two tosses and let X_2 be the number of heads in the second and third tosses.
(3) X_1 and X_2 are the first and second card (respectively) in a shuffled deck of cards.

Exercise 17 (*). Let X_1, X_2 be independent random variables with $X_1 \sim \text{Gamma}(\nu_1, \lambda)$ and $X_2 \sim \text{Gamma}(\nu_2, \lambda)$ (not that the scale parameter is the same for both).

(1) Show that $X_1 + X_2 \sim \text{Gamma}(\nu_1 + \nu_2, \lambda)$.
(2) Show that $\frac{X_1}{X_1+X_2} \sim \text{Beta}(\nu_1, \nu_2)$.
(3) Show that $X_1 + X_2$ is independent of $\frac{X_1}{X_1+X_2}$.

[**Hint:** Try to solve all three parts in one shot].

Exercise 18 (*). (1) Let X_1, X_2 be independent and have Bin(n,p) and Bin(m,p) distributions, respectively (observe that p is the same for both). Then, show that $X_1 + X_2 \sim \text{Bin}(m+n, p)$. In particular, deduce that if $\xi_1, \xi_2, \ldots, \xi_n$ are i.i.d Ber(p) random variables, then $\xi_1 + \ldots + \xi_n \sim \text{Bin}(n, p)$.
(2) Let X_1, X_2 be independent and have Pois(λ) and Pois(μ) distributions, respectively. Then, show that $X_1 + X_2 \sim \text{Pois}(\lambda + \mu)$.

[**Remark:** Some of these properties are intuitively obvious when we think of the appropriate random experiment. For example, toss a coin $m+n$ times. If X_1 is the number of heads in the first m tosses and if X_2 is the number of heads in the next n tosses, then clearly X_1, X_2 are independent with Bin(n,p) and Bin(m,p) distributions, respectively and $X_1 + X_2$ has Bin(m+n,p) distribution. This is not accepted as a proof and you must solve it by manipulating the p.m.f of X_1 and X_2].

Exercise 19. If $U \sim \text{Unif}([0,1])$ find the distribution (enough to find the density if it exists) of

(1) $Y = aU + b$ where $a > 0$ and $b \in \mathbb{R}$.
(2) $1 - U$.
(3) $-\log U$.
(4) U^m where $m \geq 1$ is an integer.

Exercise 20. Let X and Y be independent random variables with densities f_1 and f_2. In class we showed that $X_1 + X_2$ has density $g(u) = \int_{\mathbb{R}} f_1(t) f_2(u-t) dt$. Follow the techniques there to show that $X_1 X_2$ has density $h(u) = \int_{\mathbb{R}} f_1(t) f_2(u/t) \frac{1}{t} dt$.

[**Remark:** Intuitively, for $X_1]X_2$ to take the value u, X_1 can take any value t and X_2 take the value u/t. That is why the term $f_1(t) f_2(u/t)$ inside the integral is understandable. The factor of $1/t$ comes from the Jacobian determinant and is less obvious at first sight].