Problem set 7

Due date: **Not for submission**

Exercise 42. (1) Let $\xi_0, \xi_1, \xi_2, \dots$ be i.i.d. Ber(p) random variables. Fix an integer $k \geq 1$. Define $X_n = (\xi_n, \xi_{n+1}, \dots, \xi_{n+k})$ for each $n \geq 0$. Show that $X_0, X_1, \dots$ is a Markov chain and give its state space and transition probabilities.

(2) We give a verbal description of the *Birth-death chain*. Give the state space and transition probabilities. At each moment, a finite population either stays the same, increases by one, or decreases by one, with certain probabilities (that may depend on the current size of the population). Note that here if the population size is zero, it can still be reborn, and increase the size to one.

Exercise 43. (1) Let $G = (V, E)$ be the graph with vertex set $V = \{a + b\omega + c\omega^2 : a, b, c \in \mathbb{Z}\}$ and edges $u \sim v$ if and only if $v - u \in \{\pm 1, \pm\omega, \pm\omega^2\}$. Show that all states are recurrent for SRW on this graph.

(2) *Renewal chain:* Let $S = \{0, 1, 2, \dots\}$ with $p_{i,i+1} = \alpha_i$ and $p_{i,0} = 1 - \alpha_i$ for each $i \geq 0$. Assume that $0 < \alpha_i < 1$ for each i. Show that the MC is irreducible and show that it is recurrent iff and only if $\prod_{i=1}^{n} \alpha_i \to 0$ as $n \to \infty$.

Exercise 44. *Metropolis-Hasting's chain* Let S be a finite set. Let μ be a probability on S, ie., $\mu_i \geq 0$ and $\sum_i \mu_i = 1$. We will construct a Markov chain whose stationary distribution is μ.

Let $P = (p_{i,j})_{i,j \in S}$ be any transition matrix such that the corresponding Markov chain is irreducible. Define a new transition matrix $Q = (q_{i,j})_{i,j \in S}$ as

$$q_{i,j} = \begin{cases} p_{i,j}\left(\frac{\mu_j}{\mu_i} \wedge 1\right) & \text{if } j \neq i. \\ 1 - \sum\limits_{j : j \neq i} p_{i,j}\left(\frac{\mu_j}{\mu_i} \wedge 1\right) & \text{if } j = i. \end{cases}$$

Here is the interpretation of the Q-chain. If the current state is i, try to take a step according to the P-transition matrix - in other words, pick j with probability $p_{i,j}$. If $\mu_j \geq \mu_i$, then accept the offer and go to j. If $\mu_j < \mu_i$, then accept the offer with probability μ_j/μ_i.

Show that the Q chain is also irreducible and has stationary distribution μ.

[**Note:** Why care? There are many probabily distributions, even on finite sets, from which it is difficult to draw a sample. In such cases, one runs the above Markov chain for a large number of steps, and hope that the distribution of X_n is sufficiently close to π so that we can use it as a sample from π].

Exercise 45. (1) Let $P = (p_{i,j})_{i,j \in S}$ and $Q = (q_{i,j})_{i,j \in S}$ be two transition matrices on the same state space S. Suppose $\mu = (\mu_i)_{i \in S}$ satisfies $\mu_i \geq 0$ and $\mu_i p_{i,j} = \mu_j q_{j,i}$ for all $i, j \in S$. Then show that μ is a stationary measure for both P and Q (i.e., stationary measure for the corresponding Markov chains).

(2) Show that the uniform distribution on $\mathcal{S}_n$ is the unique stationary distribution for the top-to-random shuffle. One way is to construct a "random to top" shuffle and use the first part of the problem.

Exercise 46. (1) If $i \leftrightsquigarrow j$, show that either both i and j are positive recurrent or neither is.
(2) In a finite state space irreducible Markov chain, show that all states are positive recurrent.

Exercise 47. On a standard chessboard, a knight moves at random (obeying to the rules of knight moves in chess). What is the expected time for it to return to the starting square? Note that the answer depends on the starting square and you are supposed to cover all cases.

Exercise 48. (1) (*Two state chain*). Let $S = \{A, B\}$ with $P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$ where $\alpha, \beta \in [0,1]$. For what values of α, β is the chain irreducible? Find all possible stationary distributions of the chain (it is unique when the chain is irreducible).
(2) (*Renewal chain*). Let $S = \{0, 1, 2 \dots\}$ and let $p_{i,i+1} = p$, $p_{i,0} = 1 - p$, for all $i \geq 0$. Show that the chain has a unique stationary distribution and find it.

Exercise 49. One a standard chessboard, a knight is placed at a corner square. Then it starts moving at random (according to the rules of knight moves in chess). What is the expected time for it to return to the starting square? What if the starting square was not a corner but a different square (there are several cases with differing answers).

Exercise 50. Let $X_0, X_1, X_2, \dots$ be a recurrent MC. Show that $X_0, X_2, X_4, \dots$ is also a recurrent MC.